Onτ*-generalized semi compactness and τ*-generalized semi connectedness in Topological spaces

C.Aruna, Research Scholar, Research Centre of Mathematics,
Sri Parasakthi College for Women, Courtallam-627802
E Mail – arunarobert89@gmail.com
R.Selvi, Sri ParasakthiCollege for Women, Courtallam-627802

Abstract
The aim of this paper is to introduce two new classes of spaces, called τ*-gs compactandτ*-gsconnected spaces using τ*-gsclosed set and study some of their properties.

***Keywords**: τ*-gscompact space, τ*-gsconnected space, τ*-gsclosed set.*

I. Introduction

Di Miao and Noiri defined and studied about a class of compact space called s-closed space using semi open cover. Balachandranet.al. introduced a class of compact spaces called GO-compact and GO-connected spaces using g-open cover. Recently, the notion of τ*-generalized semi closed (briefly, τ*-gsclosed) set was introduced. The aim of this paper is to introduce and investigate the notions of τ*-gscompact and τ*-gsconnected utilizing τ*-gsclosed set.

II. Preliminaries

Throughout this paper (X,τ^*_1) and (Y,τ^*_2) (or simply X and Y) denote topological spaces on which on separation axioms are assumed unless explicitly stated. If A is any subset of space X, then CI(A) and Int (A) denote the closure of A and the interior of A in X respectively.

The following definitions are useful in the sequel:

Definition 2.1:A subset A of space X is called
(i) a semi-open set [5]if $A \subseteq Cl(Int(A))$
(ii) a semi –closed set [3] if $Int(Cl(A)) \subseteq A$.

Definition 2.2:[9] A subset A of a topological space X is called τ*- generalized semi closed (briefly, τ*-gsclosed) if $Cl^*_s(A) \subseteq U$, whenever $A \subseteq U$ and U is τ*- open in X. The complement of τ*-gs- closed set is τ*-generalized semi open (briefly, τ*-gsopen).
We denote the family of τ*-gs- closed sets of X by τ*-GSC(X,τ*) and τ*-gs- open sets by τ*-GSO(X,τ*)

Definition 2.3:[10] A space X is called $T_{\tau^*\text{-}gs}$-space if every τ*-gsclosed set in it is closed set.

Definition 2.4 [10] : A function $f : X \rightarrow Y$ is called
(i) τ*-generalized semicontinuous (in briefly, τ*-gscontinuous), if $f^{-1}(F)$ is τ*-gsclosed set in X for every closed set F of Y
(ii) τ*-generalized semi irresolute (in briefly, τ*-gsirresolute), if $f^{-1}(F)$ is τ*-gsclosed set in X for every τ*-gsclosed set F of Y.

III. τ*-Generalized Semi-Compactness

Definition 3.1: A collection $\{A_i : i \in I\}$of τ*-gs-open sets in a topological space is called τ*-gsopen cover of subsetA in X if $A \subseteq \cup \{A_i : i \in I\}$.

Definition 3.2:A topological space X is calledτ*-gs compact if every τ*-gsopen cover of X has a finite sub-cover.

Definition 3.3 : A subset A of topological space X is called τ*-gscompact relative to X if for every collection $\{A_i : i \in I\}$ of τ*-gsopen subset of X such that $A \subseteq \cup \{A_i : i \in I\}$there exists a finite subset $I_0$of I such that $A \subseteq \cup \{A_i : i \in I_o\}$.

Definition 3.4: A subset A of a topological space X is called τ*-gscompact if A is τ*-gscompact as a subspace of X.

Theorem 3.5: A τ*-gs-closed subset of τ*-gscompact space is τ*-gscompact relative to X.

Proof: Let A be a τ*-gsclosed subset of X. Then A^c is τ*-gsopen cover of A by τ*-gsopen subsets in X. Then $S^* = S \cup A^c$ is a τ*-gsopen cover of X. That is, $X = [\cup\{A_i : i \in I\}]\cup A^c$. By hypothesis, X is τ*-gscompact and hence S* is reducible to finite sub-cover of X say $X = A_{i1}\cup A_{i2}\cup\ldots\cup A_{in}\cup A^c, A_{ik}\in S^*$. But A and A^c are disjoint. Hence $A \subseteq A_{i1}\cup A_{i2}\cup\ldots\cup A_{in}\in S$. Thus a τ*-gsopen cover of A contains a finite sub-cover. This is, A is τ*-gscompact relative to X.

Theorem 3.6: Let $f : X \rightarrow Y$ be surjective, τ*-gscontinuous function. If X is τ*-gscompact space then Y is τ*-gscompact.

Proof: Let $\{A_i : i \in I\}$be an open cover of Y. Since f is τ*-gscontinuous function, $\{f^{-1}(A_i) : i \in I\}$ is τ*-gs-open cover of X which has a finite sub-cover say $\{f^{-1} (A_i) : i = 1,2,3, \ldots, n\}$. ThereforeX = $\cup\{f^{n-1}(A_i) : i = 1,2,\ldots,n)$ which implies $f (x) = \cup \{A_i : I = 1, 2, \ldots .n\}$ since f is surjective, $Y= \cup\{A_i : i = 1,2, \ldots n\}$.Thus $\{A_1, A_2, \ldots .,A_n\}$ is a finite sub–cover of $\{A_i : i \in I\}$ for Y. Hence Y is compact.

Theorem 3.7: If a function $f : X \rightarrow Y$ is τ*-gsirresolute and a subset of X is τ*-gscompact relative to X, then the image f(B) is τ*-gscompact relative to Y.

Proof: Let $\{A_i : i \in I\}$ be any collection of τ*-gsopen sets in Y such that $f (B) = U\{A_i : i\in I\}$. Then $B \subseteq \{f^{-1} (A_i) : i \in I\}$. Since B is τ*-gscompact relative to X, there exists finite sub collection $\{A_1, A_2,\ldots,An\}$ such that $B \subseteq \cup \{f^{-1} (A_i) : i \in I\}$. Therefore $B \subseteq \cup \{A_i : i \in I\}$. Hence f (B) is τ*-gscompact relative to Y.

Definition 3.8: If f: X→Y is said to be strongly τ*-gs open (or strongly τ*-gs closed) if image of τ*-gs open (τ*-gs closed) set of X is τ*-gs open (τ*-gs closed) set in Y.

Definition3.9: A function $f : X \rightarrow Y$ is called strongly τ*-gscontinuous if the inverse image of every τ*-gs open set in Y is open in X.

Theorem 3.10: The image of a τ*-gs compact space under a strongly τ*-gs continuous function is τ*-gs compact.

Proof: Let $f : X \rightarrow Y$ be strongly τ*-gscontinuous from a compact space X into a topological space Y. Let $\{A_i : i \in I\}$ be a τ*-gs open cover of Y. Then $\{f^{-1} (A_i) : i \in I\}$is open cover of X as f is strongly τ*-gscontinuous and so $\{f^{-1} (A_i) : i \in I\}$ is τ*-gs open cover of X. Since X is τ*-gs is compact, the τ*-gs open cover $\{f^{-1} (A_i) : i \in I\}$ of X has a finite sub cover say $\{f^{-1} (A_i) : i=1,2,\ldots.,n\}$. Therefore $X = \cup\{f^{-1} (A_i) : i=1,2,\ldots.,n\}$ which implies $f(X) = A \cup \{A_i : i=1,2,\ldots.,n\}$. Thus $Y = \cup \{A_i : i=1,2,\ldots.,n\}$.That is $\{A_1, A_2,\ldots,An\}$ is a finite sub cover of $\{A_i : i \in I\}$for Y. Hence Y is τ*-gs compact.

Theorem 3.11: If $f : X \rightarrow Y$ be strongly τ*-gs from a compact space X into a topological space Y, then Y is τ*-gs compact.

Proof: Let$\{A_i : i \in I\}$ be a τ*-gs open cover of Y. Since f is strongly τ*-gscontinuous$\{f^{-1} (A_i) : i \in I\}$ is open cover of X. Again since X is compact space , the open cover $\{f^{-1} (A_i) : i \in I\}$ of X has a finite sub cover say$\{f^{-1} (A_i) : i=1,2,\ldots.,n\}$. Therefore $X = \cup\{f^{-1} (A_i) : i=1,2,\ldots.,n\}$ which implies $f(X) = \cup \{A_i : i=1,2,\ldots.,n\}$,so that $Y = \cup \{A_i : i=1,2,\ldots.,n\}$. That is $\{A_1, A_2\ldots An\}$ is a finite sub cover of $\{A_i : i \in I\}$ for Y. Hence Y is τ*-gs compact.

Theorem 3.12: If f : X →Y be τ*-gs-continuous function from a compact space X into a topological space Y. If Y is $T_{\tau^*\text{-}gs}$-space, then Y is τ*-gscompact.

Proof: Let {Ai :i ∈ I} be a τ*-gs open cover of Y. As Y is $T_{\tau^*\text{-}gs}$space, {A_i : i ∈ I} open cover of Y. Since f is τ*-gscontinuous, {f^{-1} (A_i) :i ∈ I} is τ*-gs open cover of X. Again since X is τ*-gs compact space, the τ*-gsopen cover {f^{-1}(A_i) :i ∈ I} of X has a finite sub-cover say :{f^{-1} (Ai) :i =1, 2, 3, ….,n}. Therefore X = ∪{f^{-1} (A_i) :i=1,2,….,n} which implies f (X) = ∪ {A_i : i=1,2,,..n} so that Y = ∪ {A_i : i=1,2,,..n}. That is {A_1, A_2, ….., An} is a finite sub–cover of {Ai : i ∈ I} for Y. Hence Y is τ*-gs compact.

Theorem3.13: Every τ*-gs compact space is compact.

Proof: Let X be a τ*-gs compact space. Let {A_i :i ∈ I} is τ*-gsopen cover of X. Then {A_i :i ∈ I} is a τ*-gsopen cover of X as every open set is τ*-gs open set. Since X is τ*-gscompact, the τ*-gsopen cover {A_i :i ∈ I} of X has a finite sub–cover say {A_i : i = 1,2,3,….n} for X. Hence X is compact.

Theorem 3.14:If X is compact and $T_{\tau^*\text{-}gs}$ space, then X is τ*-gs compact.

Proof: Let {A_i :i ∈ I} be a τ*-gs-open cover of X. As X is $T_{\tau^*\text{-}gs}$– space,{A_i :i ∈ I} open cover of X. Since X is compact, the open cover {A_i :i ∈ I} of X has a finite sub cover say {A_i :i = 1, 2. 3, …,n}. Hence X is τ*-gs compact.

Theorem 3.15: A topological space X is τ*-gs compact if and only if every family of τ*-gsclosed sets of X having finite intersection property has a nonempty intersection.

Proof: Suppose X is τ*-gs compact. Let {A_i :i ∈ I} be a family of τ*-gsclosed sets with finite intersection property. To prove that ∩ {A_i :i ∈ I}≠ ∅. Suppose ∩ {A_i :i ∈ I}= ∅. Then X – ∪{A_i :i ∈ I}= X, which implies {(X – A_i) : i ∈ I}=X. Thus the cover {X – A_i :i ∈ I} is a τ*-gs open cover of X. Since X is τ*-gscompact, the τ*-gsopen cover {X – A_i : i ∈ I} has a finite sub – cover say {(X – A_i) : i = 1, 2, …., n). This implies X = ∪ {(X-Ai) : i = 1, 2, ….n} .Hence X – X = X – [X - ∩{A_i :i =1, 2, …,n}] implies that ∅ = ∩ {Ai : i = 1, 2, …,n}. This contradicts the assumption. Hence ∩ {A_i :i ∈ I}≠ ∅.

Conversely, suppose every family of τ*-gsclosed sets of X with finite intersection property has a non-empty intersection. To prove that X is τ*-gscompact. Suppose X is not τ*-gscompact. Then there exists a τ*-gsopen cover of X say {G_i :i∈ I} having no finite sub cover. This implies for any finite sub family {G_i :i=1,2,…, n} of {G_i : i ∈ I} we have ∪ {G_i :i =1, 2, ….,n} ≠X which implies that X – ∪ {G_i : i = 1,2,…,n} ≠ X – X, which implies ∩{(X-Gi) i=1, 2, …,n} ≠ ∅. Then the family {X-G_i : i ∈ I} of τ*-gsclosed sets has finite intersection property. Also by assumption ∩{(X-G_i) : i ∈ I}≠ ∅ which implies X – ∪ {G_i :i ∈ I} ≠ ∅. So ∪ {G_i : i ∈ I}≠X. This implies {G_i :i∈ I} is not a cover of X. This contradicts the fact that {G_i : i ∈ I} is cover of X. Thus a τ*-gsopen cover{G_i : i ∈ I} has a finite sub – cover {G_i : i = 1,2,…,n}. Hence X is τ*-gscompact.

IV. τ*-Generalized Semi Connectedness

We define and study the concept of τ*-gsconnectedness in the following:

Definition 4.1: A topological space X is said to be τ*-gsconnected if X cannot be written as disjoint union of two non emptyτ*-gsopen sets.

A subset of X is τ*-gsconnected if it is τ*-gsconnected as subspace.

Theorem 4.2: for a topological space X, the following are equivalent.

(i) X is τ*-gsconnected

(ii) X and ∅ are the only subset of X which are both τ*-gsopen and τ*-gsclosed.

(iii) Each τ*-gscontinuous function of X into a discrete space Y with at least two points is a constant function.

Proof: (i) → (ii). Let U be τ*-gsopen and τ*-gsclosed subset of X. Then X – U is both τ*-gsopen and τ*-gsclosed. Since X is the disjoint union of the τ*-gsopen sets U and X – U, one of these must be empty, that is U = ∅ or U = X.

(ii) →(i). Suppose that X = A ∪ B where A and B are disjoint non– empty τ*-gsopen subsets of X. Then A is both τ*-gsopen and τ*-gsclosed. By assumption A = ∅ or A = X. Therefore A is τ*-gsconnected.

(ii) → (iii).Let f : X →Y be τ*-gscontinuous function then X is covered by τ*-gs open and τ*-gsclosed covering $\{f^{-1}(y) : y \in Y\}$, By assumption $f^{-1}(y) = \emptyset$ or X for each y ∈ Y. If $f^{-1}(y) = \emptyset$ for all y ∈ Y, then f is fails to be function. Then, there exists only one point y ∈ Y such that $f^{-1}(y) \neq \emptyset$ and hence $f^{-1}(y) = X$. This shows that f is a constant function.

(iii) → (ii). Let U be both τ*-gsopen and τ*-gsclosed in X. Suppose U ≠ ∅. Let f : X → Y be τ*-gs-continuous function defined by f(U) = {y} and f (X-U) = {w} for some distinct y and w in Y. By assumption f is constant. Therefore we have U = X.

Theorem 4.3:

(i) Iff: X →Y be τ*-gscontinuous surjection and X is τ*-gsconnected, then Y is connected.

(ii) Iff: X →Y be τ*-gs irresolute surjection and X is τ*-gsconnected, then Y is τ*-gs connected.

Proof: (i) Suppose that X is not connected, Let Y = A∪B where A and B are nonempty open sets in Y. Since f is τ*-gscontinuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty and τ*-gsopen in X. This contradicts the fact that X is τ*-gsconnected. Hence Y is connected.

(ii) The argument is a minor modification of the proof (i).

Theorem 4.4: Suppose that X is a $T_{\tau^*\text{-}gs}$- space then X is connected if and only if it is τ*-gs connected.

Proof: Suppose that X is connected. X cannot be expressed as disjoint union of two nonempty proper subsets of X. Suppose X is not τ*-gsconnected. Let A and B are any two τ*-gsopen subsets of X such that X = A∪B, where A∩B= ∅and A ⊂ X, B ⊂ X. Since X is $T_{\tau^*\text{-}gs}$-space and A, B are τ*-gsopen sets, A, B are open subsets of X, which contradicts the fact that X is connected. Therefore X is τ*-gs connected.

Conversely, every open set is τ*-gsopen set. Therefore every connected space is τ*-gsconnected.

References

[1] BalachandranK.,Sundaram P. and Maki H., on generalized continuous maps in topological space,Mem.Fac. Sci. Kochi. Uni. Ser.A (Math.), 12 (1991), 5-13.

[2] CrosselyS.G.andHildbrand S.K., On semi- closure. Texas J.Sci, 22 (1971), 99-112.

[3] Di Maio G., Noiri T., On s-closed space, Indian J.Pure Appl. Math., 18 (1987), 226-233

[4] DunhamW., A new closure operator for non-T_1 topologies, Kyungpook Math.J.22(1982),55-60

[5] Levine N., Semi-open sets and semi –continuity in topological spaces, Amer.Math., Monthly, 70 (1963), 36-41.

[6] GovindappaNavalagi and paga Md. Hanif, on θ-gs-Neighbiurhoods, Inidan journal of Mathematics and Mathematical Sciences, 4(1) (June 2008), 21-31.

[7] GovindappaNavalagi and Paga Md. Hanif, on θ-gsContinuity andθ-gs- Irresoluteness, International journal of Mathematics and computer Science and Technology, 1(1) (January – June), 95-101.

[8] PushpalathaA.S.Eswaran and P.RajaRubi, -generalized closed sets in topological spaces, WCE 2009, July 1-3, 2009, London, U.K.

[9] C.Aruna, R. Selvi, τ*- generalized semi closed sets, International journal of Engineering Sciences and Research, ISSN: 2277 – 9655.

[10] C.Aruna & R.Selvi, contra τ*-generalized semi continuity (communicated).