# On the Cubic Equation with Four Unknowns

$$2x^3+8z^3=2y^3+8w^3+12(x-y)^3$$

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**Abstract**
The cubic equation $2x^3+8z^3=2y^3+8w^3+12(x-y)^3$ is analyzed for its non – zero distinct integer solutions. Five different patterns of non-zero distinct integer solutions to the equation under consideration are obtained. A few interesting relation between the solutions and special numbers are exhibited.

***Keywords:*** *Integral solutions, Ternary Cubic.*

## I. INTRODUCTION

The Cubic Equation offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems, one may refer [3-10]. This communication concerns with yet another interesting Ternary Quadratic equation $2x^3+8z^3=2y^3+8w^3+12(x-y)^3$ representing a homogenous cone for determining its infinitely may non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

## II. NOTATIONS

obl $_n$- Oblong number of rank 'n'
$t_{m,n}$- Polygonal number of rank 'n' with sides'm

## III. METHOD OF ANALYSIS

The cubic Diophantine equation with four unknowns to be solved for getting non – zero integral solution is

$$2x^3+8z^3=2y^3+8w^3+12(x-y)^3 \quad \text{---- } (1)$$

On substituting the linear transformations

$$x=u+v,\ y=u-v,\ w=s+v,\ z=s-v \quad \text{-}(2)$$

In (1), we get

$$12u^2v=48s^2v+6v^3+96v^3+6v^3$$

$$u^2=4s^2+9v^2 \quad \text{------- } (3)$$

Which is in the form of famous Pythagorean equation.

We obtain five different patterns of integral solutions to (1) through solving (3) which are illustrated as follows:

***Pattern 1:***

In equation (3), which is satisfied by

$$v=\frac{2}{3}pq$$

$$s=\frac{p^2-q^2}{2}$$

Put $p = 6P$ and $q = 6Q$ in $v, s, u$ we get

$$u=p^2+q^2$$

$$=36(P^2+q^2)$$

$$v=\frac{2}{3}pq$$

$$v=\frac{2}{3}(6p)(6q)$$

$$= 24PQ$$

$$s = \frac{p^2 - q^2}{2}$$

$$= 18(p^2 - q^2)$$

$$x = u + v$$

$$= 36p^2 + 36q^2 + 24pq$$

$$= 36(p^2 + q^2) + 24pq$$

$$y = u - v$$

$$= 36p^2 + 36q^2 - 24pq$$

$$= 36(p^2 + q^2) - 24pq$$

$$z = s - v$$

$$= 18p^2 - 18q^2 - 24pq$$

$$= 18(p^2 - q^2) - 24pq$$

$$w = s + v$$

$$= 18p^2 - 18q^2 + 24pq$$

$$= 18(p^2 - q^2) + 24pq$$

Now, we get

$$x = x(p,q) = 36(p^2 + q^2) + 24pq$$

$$y = y(p,q) = 36(p^2 + q^2) - 24pq$$

$$z = z(p,q) = 18(p^2 - q^2) - 24pq$$

$$w = w(p,q) = 18(p^2 - q^2) + 24pq$$

**Properties**

(i) $3[x(p,q) + y(p,q)]$ is a nasty number
(ii) $[x(p,q) + z(p,q)]$ is a nasty number
(iii) $[x(p,1) + w(p,1)] - 60t_{20,p} \equiv 0 \pmod 2$
(iv) $[y(p,q) + w(p,q)]$ is a nasty number
(v) $[x(\alpha^2 + 1) + w(\alpha^2+1)] - 6t_{20,\alpha^2}$ is a nasty number

***pattern 2 :***

***Let***

$$2s = 2pq \Rightarrow s = pq$$

$$v = \frac{p^2 - q^2}{3} \text{ and } u = p^2 + q^2$$

*Put* $p = 3p$ *and* $q = 3q$*, we get*

$$u = 3(p^2 + q^2)$$

$$v = (p^2 - q^2)$$

$$s = 9pq$$

In view of (2) the non- zero integer solutions to (1) are given by

$$x = x(p,q) = 4p^2 + 2q^2$$

$$y = y(p,q) = 2p^2 - 2q^2$$

$$z = z(p,q) = 9pq - p^2 + q^2$$

$$w = w(p,q) = p^2 - q^2 + 9pq$$

**Properties**

(i) $[x(p,q) + y(p,q)]$ is a nasty number
(ii) $[z(p,q) + w(p,q)]$ is a nasty number
(iii) $[x(p,1) + z(p,1)] \equiv 0 \pmod 3$
(iv) $[y(p,1) - z(p,1)]$ is a nasty number
(v) $[y(p,1) - z(p,1)] \equiv 0 \pmod 3$

***pattern 3 :***

Equation (3) can be re – written as

$$u^2 - (3v)^2 = (2s)^2$$

which is written in the form of ratio as,

$$\frac{u+3v}{2s}=\frac{2s}{u-3v}=\frac{\alpha}{\beta}\text{---(4)}$$

Which is equivalent to the system of equations,

$u\beta + 3v\beta - 2s\alpha = 0$

$-u\alpha + 3\alpha v + 2\beta s = 0$

Applying the method of cross multiplication we have,

$$u = 6\alpha^2 + 6\beta^2$$

$$v = 2\alpha^2 - 2\beta^2$$

$$s = 6\alpha\beta$$

In view of (2) the non – zero integer solutions to (1) are given by

$$x = x(\alpha,\beta) = 8\alpha^2 + 4\beta^2$$

$$y = y(\alpha,\beta) = 4\alpha^2 + 8\beta^2$$

$$w(\alpha,\beta) = 2\alpha^2 - 2\beta^2 + 6\alpha\beta$$

$$z = z(\alpha,\beta) = 2\beta^2 - 2\alpha^2 + 6\alpha\beta$$

**Properties**

$[x(\alpha,\beta) + y(\alpha,\beta)]$ is a nasty number
(ii) $[x(\alpha,1) + w(\alpha,1)] \equiv 0 (\mathrm{mod}\, 2)$
(iii) $[x(\alpha,\beta) - w(\alpha,\beta)]$ is a nasty number
(iv) $[w(\alpha,1) - z(\alpha,1)] \equiv 0 (\mathrm{mod}\, 4)$
(v) $y(\alpha,1) + z(\alpha,1)] \equiv 0 (\mathrm{mod}\, 2)$

***pattern 4:***

Equation (3) can be re – written as

$$u^2 - (2s)^2 = (3v)^2$$

Which is equivalent to the system of equations,

$u\beta + 2s\beta - 3\alpha v = 0$

$-u\alpha + 2s\alpha + 3v\beta = 0$

Applying the method of cross multiplication we have,

$$u = 6\alpha^2 + 6\beta^2$$

$$s = 3\alpha^2 - 3\beta^2$$

$$v = 4\alpha\beta$$

In view of (2) the non – zero integer solutions to (1) are given by

$$x = x(\alpha,\beta) = 6\alpha^2 + 6\beta^2 + 4\alpha\beta$$

$$y = y(\alpha,\beta) = 6\alpha^2 + 6\beta^2 - 4\alpha\beta$$

$$w = w(\alpha,\beta) = 3\alpha^2 - 3\beta^2 + 4\alpha\beta$$

$$z = z(\alpha,\beta) = 3\alpha^2 - 3\beta^2 - 4\alpha\beta$$

**Properties**

(i) $x(\alpha,1) + y(\alpha,1) - 12$ is a nasty number
(ii) $[x(\alpha,1) + y(\alpha,1)] \equiv 0 (\mathrm{mod}\, 2)$
(iii) $[w(\alpha,\beta) + z(\alpha,\beta)] \equiv 0 (\mathrm{mod}\, 3)$
(iv) $[y(\alpha,1) + 2z(\alpha,1)] \equiv 0 (\mathrm{mod}\, 4)$
(v) $x(\alpha,1) + z(\alpha,1) \equiv 0 (\mathrm{mod}\, 3)$

***pattern 5:***

Equation (5) can also be equivalent to the system of equations,

$u\beta + 2s\beta - \alpha v = 0$

$-u\alpha + 2s\alpha + 9v\beta = 0$

Applying the method of cross multiplication we have,

$$u = 2\alpha^2 + 18\beta^2$$

$$s = \alpha^2 - 9\beta^2$$

$$v = 4\alpha\beta$$

In view of (2) the non – zero integer solutions to (1) are given by

$$x = x(\alpha,\beta) = 2\alpha^2 + 18\beta^2 + 4\alpha\beta$$

$$y = y(\alpha,\beta) = 2\alpha^2 + 18\beta^2 - 4\alpha\beta$$

$w = w(\alpha,\beta) = \alpha^2 - 9\beta^2 + 4\alpha\beta$

**Properties**

(i) $[x(\alpha,1) - y(\alpha,1)] \equiv 0 \pmod 8$

(ii) $[x(\alpha,1) + w(\alpha,1)] \equiv 0 \pmod 3$

(iii) $6x(\alpha,1) - 3w(\alpha,1)] \equiv 0 \pmod 3$

(iv) $[w(\alpha,1) - z(\alpha,1)] \equiv 0 \pmod 8$

(v) $[y(\alpha,1) + w(\alpha,1)] \equiv 0 \pmod 3$

## IV. CONCLUSION

To conclude, one may search for other patterns of solutions to the equation under consideration.

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