On the Ternary Cubic Diophantine Equation $x^3 + y^3 = 2(z+w)^2(z-w)$

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Abstract

The cubic equation $x^3 + y^3 = 2(z+w)^2(z-w)$ is analyzed for its non – zero distinct integer solutions. Three different patterns of non-zero distinct integer solutions to the equation under consideration are obtained. A few interesting relation between the solutions and special numbers are exhibited.

Keywords: *Integral solutions, Ternary Cubic.*

I. INTRODUCTION

The Cubic Equation offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems, one may refer [3-10]. This communication concerns with yet another interesting Ternary Quadratic equation $x^3 + y^3 = 2(z+w)^2(z-w)$ representing a homogenous cone for determining its infinitely may non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

II. NOTATIONS

obl_n- Oblong number of rank 'n'
$t_{m,n}$- Polygonal number of rank 'n' with sides'm

III. METHOD OF ANALYSIS

The ternary cubic equation with four unknowns to be solved for getting non – zero integral solution is

$$x^3 + y^3 = 2(z+w)^2(z-w) \text{ ----(1)}$$

On substituting the linear transformations

$$x = u+v\ ,\ y = u-v\ ,\ z = u+p\ ,\ w = u-p,$$
$$u \neq p \ and \ \ v \neq p \ \ ----(2)$$

Then

$x^3 + y^3 = 2(z+w)^2(z-w)$ becomes

$$(u+v)^3 + (u-v)^3 = 2[(u+p)+(u-p)]^2[(u+p)-(u-p)]$$
$$2u(u^2+3v^2) = 8u^2 2p$$

$$(u^2+3v^2) = 8up$$

It can be written as

$$(u-4p)^2 + 3v^2 = 16p^2 \text{ ------ (3)}$$

We obtain three different patterns of integral solutions to (1) through solving (3) which are illustrated as follows :

1. Pattern 1:

In (3) take**----------**(4a)

and write '16' as

$$16 = (2+2i\sqrt{3})(2-2i\sqrt{3}) \text{ -------- (4b)}$$

Substituting (4a) and (4b) in (3) and employing the method of factorization,

$$(u-4p)^2 + 3v^2 = (2+2i\sqrt{3})(2-2i\sqrt{3})(a^2+3b^2)^2$$

Which is equivalent to the system of equations

$$(u-4p+i\sqrt{3}v) = (2+2i\sqrt{3})(a+i\sqrt{3}b)^2$$

$$(u-4p-i\sqrt{3}v) = (2-2i\sqrt{3})(a-i\sqrt{3}b)^2$$

Equating the real and imaginary parts, we have

$$u-4p = 2a^2 - 6b^2 - 12ab \text{ ----- (5)}$$

$$v = 2a^2 - 6b^2 + 4ab \text{ ----------(6)}$$

substituting (4a) in (5), we get

$u-4(a^2+3b^2)=2a^2-6b^2-12ab$

$u=6a^2+6b^2-12ab$ -------- (7)

From (4a), (6), (7) and (2), we get

$x=u+v=8(a^2-ab)$

$y=u-v=4(a^2+3b^2-4ab)$

$z=u+p=7a^2+9b^2-12ab$

$w=u-p=5a^2+3b^2-12ab$

The distinct integral solutions of (1) are expressed by,

$x=x(a,b)=8(a^2-ab)$

$y=y(a.b)=4(a^2+3b^2-4ab)$

$z=z(a,b)=7a^2+9b^2-12ab$

$w=w(a,b)=5a^2+3b^2-12ab$

1.1 Properties:

(i) $x(1,b)+y(1,b)$ is a nasty number
(ii) $z(1,b)+w(a,b)\equiv 0(\text{mod}\,2)$
(iii) $z(a,b)+w(a,b)\equiv 0(\text{mod}\,2)$
(iv) $y(a,b)+w(a,b)\equiv 0(\text{mod}\,3)$
(v) $3x(a,1)+4z(a,b)\equiv 0(\text{mod}\,3)$

2. Pattern 2:

Taking $u-4p=4X$, $v=4V$ ----- (8)

in (3) , we have

$X^2+3V^2=p^2$---------(8a)

Which is satisfied by

$X=a^2-3b^2, V=2ab, P=a^2+3b^2$ --(9)

On substituting, we get

$u-4p=4X$

$u=4X+4p$

$=4(a^2-3b^2)+4(a^2+3b^2)$

$=8a^2$ ---------(10a)

$v=4V$

$v=4V$

$=4(2ab)=8ab$ ----- (10b)

From (9) (10a) (10b) in (2), we get

$x=u+v$

$=8(a^2+ab)$

$y=u-v$

$=8(a^2-ab)$

$z=u+p$

$=3(3a^2+b^2)$

$w=u-p$

$=7a^2-3b^2$

The distinct integral solutions of (1) are given by,

$x=x(a,b)=8(a^2+ab)$

$y=y(a,b)=8(a^2-ab)$

$z=z(a,b)=3(3a^2+b^2)$

$w=w(a,b)=7a^2-3b^2$

2.1 Properties:

(i) $x(a,b)+y(a,b)\equiv 0\ (\text{mod}\,8)$

(ii) $z(a,1)+w(a,1)\equiv 0(\text{mod}\,4)$
(iii) $y(a,1)+z(a,1)-t_{36,a}\equiv 0(\text{mod}\,8)$
(iv) $y(a,1)+w(a,1)-t_{32,a}\equiv 0(\text{mod}\,3)$
(v) $x(a,1)+2z(a,1)-t_{36,a}\equiv 0(\text{mod}\,3)$
(vi) $3z(a,b)-w(a,b)\equiv 0(\text{mod}\,2)$

3. Pattern 3:

Rewrite (8a) as

$$P^2 - 3V^2 = X^2 * 1 \text{------}(11)$$

and write '1' as

$$1 = (2+\sqrt{3})(2-\sqrt{3}) \ \& \ X = a^2 - 3b^2 \text{-- } (12)$$

Substituting (12) in (11) and using method of factorization , define

$$p^2 - 3V^2 = (a^2 - 3b^2)^2 * (2+\sqrt{3})(2-\sqrt{3})$$

$$(p+\sqrt{3})(p-\sqrt{3})V = (a+\sqrt{3}b)^2 (a-\sqrt{3}b)^2 (2+\sqrt{3})(2-\sqrt{3})$$

Equating rational and irrational parts on both sides , we obtain

$$(p+\sqrt{3}) = (a+\sqrt{3}b)^2 (2+\sqrt{3})$$

$$(p-\sqrt{3}) = (a-\sqrt{3}b)^2 (2-\sqrt{3})$$

Hence

$$p = 2a^2 + 6b^2 + 6ab \text{------ } (13)$$

$$v = a^2 + 3b^2 + 4ab \text{------- } (14)$$

From (8) (13) (14), we have

$$u - 4p = 4X$$

$$u = 4X + 4p$$

$$= 12a^2 + 12b^2 + 24ab \text{----- } (15)$$

$$v = 4V$$

$$= 4a^2 + 12b^2 + 16ab \text{------ } (16)$$

$$p = 2a^2 + 6b^2 + 6ab$$

On substituting (13), (15),(16) in (2), we get

$$x = u + v = 8(2a^2 + 3b^2 + 5ab)$$

$$y = u - v = 8(a^2 + ab)$$

$$z = u + p = 2(7a^2 + 9b^2 + 15ab)$$

$$w = u - p = 2(5a^2 + 3b^2 + 9ab)$$

The distinct integral solutions of (1) are expressed by,

$$x = x(a,b) = 8(2a^2 + 3b^2 + 5ab)$$

$$y = y(a,b) = 8(a^2 + ab)$$

$$z = z(a,b) = 2(7a^2 + 9b^2 + 15ab)$$

$$w = w(a,b) = 2(5a^2 + 3b^2 + 9ab)$$

3.1 Properties:

(i) $x(a,b) - y(a,b) \equiv 0 \pmod 8$

(ii) $x(a,1) - y(a,1) - t_{8,a} \equiv 0 \pmod 3$

(iii) $z(a,b) - w(a,b) \equiv 0 \pmod 4$

(iv) $x(a,1) - 2y(a,1) - 34a$ is a nasty number

(v) $10y(a,1) - 8w(a,1) \equiv 0 \pmod 8$

IV. CONCLUSION

To conclude, one may search for other patterns of solutions to the equation under consideration.

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