A New Approach on Triangular Fuzzy Random Variable in Mathematical Economics

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ABSTRACT

In this paper, we discuss the finite pure exchange economy concept using a triangular fuzzy random variable with its parameters mean μ and standard deviation σ respectively. Based on these, some lemma are proved.

Keywords: *Initial allocation, Finite pure exchange economy, Consumption fuzzy set, convex function and Triangular fuzzy random variables.*

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1. Introduction

Mathematical tools cannot be directly used in few areas of economy. In this case, we use the concept of fuzziness and the suitable interpretation is a convenient component of the rigorous language of theoretical economics. Mila Stojakovic [2], delivered the idea of the fuzzy random variables in the finite pure exchange economic systems. Now, we discuss the system of economics through the triangular fuzzy random variables with its parameters mean μ and standard deviation σ. Obviously, the uncertainty is the basic concept of fuzziness and it is interpreted through the triangular fuzzy random variable with its parameters mean μ and standard deviation σ, it is used for the individual characters of the agents in well - known simple economic system - finite pure exchange economy. In this paper, we show how the triangular fuzzy random variable can be used for the purpose of modelling and analysing an economic system. Economics deals with all stem from the theory of general economic equilibrium. In the theory of equilibrium analysis it is usual to define an economy or economic model as a system of sets, as a mapping. The basic definitions and notions of the triangular fuzzy random variables are used to define the basic terms such as the preferences or tastes of the economic agents.

The association of the paper is as follows. In section 2, we recall the basic definition and notions of the triangular fuzzy random variables and the new definitions of the distribution function $(\mu-\sigma, \mu)$ and survival function $(\mu,\ \mu+\sigma)$ of the sales (prices) of the agent (commodity). In section 3, we prove some lemma based on infinite pure exchange economy via the triangular fuzzy random variables with its parameters mean μ and standard deviation σ .

2. Preliminaries

Now, in this section, we discuss the basic terms and properties which can be found in [1 -5].

Definition 2.1: A fuzzy set A on the Universal set X is defined as the set ordered pairs

$A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) \in [0,1]\}$, where $\mu_A(x)$ is its membership function.

Definition 2.2: The support of fuzzy set A is the set of all points x in X such that $\mu_A(x) > 0$.

That is, Support (A) = $\{x \in X / \mu_A(x) > 0\}$.

Definition 2.3: The α-cut $A_\alpha$of a fuzzy set A is the set consisting of those elements of the universe X whose membership values exceed the threshold level $\alpha \in [0, 1]$.

That is, $A_\alpha = \{x \in X / \mu_A(x) \geq \alpha\}$

Definition 2.4: A fuzzy set A on the set R of real numbers said to be a fuzzy number, if:

i. A is a normal fuzzy set, i.e. there exists $x \in X$ such that $\mu_A(x) = 1$.

ii. A_α is a closed interval for every $\alpha \in [0,1]$, i.e. A_α is a convex subset of R.

iii. The membership function $y= \mu_A(x)$ is a piecewise continuous function.

Among the various shapes of fuzzy numbers, the triangular fuzzy number (TFN) is the most popular one. A TFN is defined as follows:

Definition 2.5:

The triangular fuzzy number is a fuzzy number represented with 3-tuples as follows: $A = (a_1, a_2, a_3)$. This representation is interpreted as membership function and holds the following conditions.

(i) a_1 to a_2 is increasing function

(ii) a_2 to a_3 is decreasing function

(iii) $a_1 \leq a_2 \leq a_3$ is a fuzzy number with membership function

$$\mu_A(x) = \begin{cases} 0 & for\ x < a_1 \\ \dfrac{x-a_1}{a_2-a_1} & for\ a_1 \leq x < a_2 \\ \dfrac{a_3-x}{a_3-a_2} & for\ a_2 \leq x < a_3 \\ 0 & for\ x \geq a_3 \end{cases}$$

It is easy to check that the α-cut of a TFN $A = (a_1, a_2, a_3)$ is of the form

$A_\alpha = [a_1^\alpha, a_3^\alpha]$ with $a_1^\alpha = (a_2 - a_1)\,\alpha + a_1$, $a_3^\alpha = -(a_3 - a_2)\alpha + a_3$

2.6 Formation of the triangular fuzzy random variables from the normal distribution curve.

In normal distribution, the structure of the normal curve becomes very sharp peak as well as symmetric with respect to the mean (μ), when the standard deviation (σ) is very minimum, so that the normal distribution curve is even.

Now, the base ($a_1=\mu-n\sigma$, $a_2=\mu$, $a_3 = \mu + n\sigma$), n = 1, 2, 3 of the normal curves shielded the area 99.73%. The remaining area 0.27% spreads over outside of the range $|X-\mu| \geq 3\sigma$ on both sides. Here, each side 0.135% of the region is shielded and that the X – axis is asymptote to the curve. The above mentioned characteristics can also exists in triangular fuzzy number, the base of the triangular fuzzy number is ($a_1 =\mu-n\sigma$, $a_2=\mu$, $a_3 = \mu+n\sigma$), n = 1, 2, 3 shielded the area is 99.73%. The remaining area 0.27% spreads over outside of the range $|X-\mu| \geq 3\sigma$ on both sides. Here, each side 0.135% of the region is shielded and that the support of the triangular fuzzy number is asymptote for that curve. Thus, the triangular fuzzy number and the normal distribution curve have the same characteristics. Hence, the triangular fuzzy number satisfies all the properties of the normal distribution.

Definition: 2.7

If X be a triangular fuzzy random variable with mean μ_1, and standard deviation σ_1 and a function f: $[\mu_1-\sigma_1, \mu_1+\sigma_1] \to [0, 1]$ is said to be concave function of the triangular fuzzy number.

Then $P\{bX_\alpha^L + (1-b)X_\alpha^U\} \geq (P\{X_\alpha^L\})^{\mathbf{b}}(P\{X_\alpha^U\})^{\mathbf{1-b}}$

$$P\{(bX_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1) \geq 0 \vee ((1-b)X_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\}$$

$\geq (P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0\})^{\mathbf{b}}(P\{(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1) \leq 0)\})^{\mathbf{1-b}}, \quad 0 \leq b \leq 1.$

Similarly, $P\{bX_\alpha^L + (1-b)X_\alpha^U\} \leq (P\{X_\alpha^L\})^{\mathbf{b}}(P\{X_\alpha^U\})^{\mathbf{1-b}}$

$$P\{(bX_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1) \geq 0 \vee ((1-b)X_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\}$$

$\leq (P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0\})^{\mathbf{b}}(P\{(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1) \leq 0)\})^{\mathbf{1-b}}, 0 \leq b \leq 1.$

is called convex function of the triangular fuzzy number.

Definition: 2.8

The probability distribution function is given by $F(x) = \begin{cases} 0 & x \leq a \\ \frac{(x-a)}{(b-a)} & a < x < b \\ 1 & x \geq b \end{cases}$

Where $f(x) = \frac{1}{(b-a)}$ is the probability density function in $a \leq x \leq b$.

Now, we construct the distribution function of the triangular fuzzy number

$$(a, b, c) = (\mu_1-\sigma_1, \mu_1, \mu_1+\sigma_1) \text{ is } \frac{(x-a)}{(b-a)} \geq \alpha$$

$\frac{X_\alpha^L-(\mu_1-\sigma_1)}{\sigma_1} \geq \alpha$

$$X_\alpha^L-(\mu_1-\sigma_1) \geq \alpha\sigma_1$$

Hence, $a_1^\alpha = \{(X_\alpha^L - (\alpha - 1)\sigma_1 - \mu_1) \geq 0\}$ which is the required distribution of the triangular fuzzy number and it is also called the membership function of the increasing function of the triangular fuzzy number.

Definition: 2.9

The complementary probability distribution or survival function is given by

$$G(x) = \begin{cases} 1 & x \leq b \\ 1 - \frac{(x-b)}{(c-b)} & b < x < c \\ 0 & x \geq c \end{cases}$$

Where $g(x) = \frac{1}{(c-b)}$ is the probability density function in $b \leq x \leq c$ and

$F(x) = 1 - G(x)$.

Now, we construct the survival functions of the triangular fuzzy number

$(a, b, c) = (\mu_1 - \sigma_1, \mu_1, \mu_1 + \sigma_1)$ is as follows:

$$1 - \frac{(x-b)}{(c-b)} = 1 - \frac{(x-\mu_1)}{(\mu_1+\sigma_1)-\mu_1} \geq \alpha$$

$$\frac{\sigma_1 - X_\alpha^L + \mu_1}{\sigma_1} \geq \alpha$$

$$\sigma_1 - X_\alpha^L + \mu_1 \geq \alpha\sigma_1$$

$$-X_\alpha^L \geq \alpha\sigma_1 - \sigma_1 - \mu_1$$

$$X_\alpha^U \leq -(\alpha - 1)\sigma_1 + \mu_1$$

$a_3^\alpha = \{(X_\alpha^U + (\alpha - 1)\sigma_1 - \mu_1) \leq 0\}$ which is the required survival function of the triangular fuzzy number and it is also called the membership function of the decreasing function of the triangular fuzzy number. So that each α - cut value is the union of the distribution and the survival function.

In this paper, we use the triangular fuzzy random variable for two concepts such as sales capacity of the agent and the prices of the commodity. The support of the first triangular fuzzy random variables refers the range of the sales effort of the agent for complete the target in amount. Generally, the sale target is fixed by company is beyond the realistic sales level in specified period, so that this amount of target has two stages, one is within $(\mu - \sigma, \mu)$ (distribution function) the realistic sales level and the other is beyond $(\mu, \mu + \sigma)$ (survival function) the realistic sales level in a specific period. Similarly, the support of the second triangular fuzzy random variables refers the range of the prices of the commodity in amount. Generally, the prices of the commodity also have two stage one is within $(\mu - \sigma, \mu)$ (distribution function) the realistic buying price level of the commodity of the people and beyond $(\mu, \mu + \sigma)$ (survival function)the realistic buying prices level of the commodity of the people. The point in which the end and beginning of the distribution and survival functions is that when $\alpha = 1$.

Now, we deal with some well - known basic definitions and notions of the finite pure exchange economic system and the relation with the triangular fuzzy random variable, which will be used in the next section.

The concept of an economy or an economic system may be formalized in different ways. When the theory is only concerned with the economic exchange system and process (the markets), with total independence of what happens in the production sector, then this is best represented by an economic system without production, called a **Pure exchange economy** and conceived as a structure $\mathcal{E} = ((A, \mathcal{A}, \mu), X, R^2)$.

The triple $(A,\mathcal{A},\mu)$ is a probability space of agents (consumers) with the following economic interpretation: A is the set of agents (consumers); $\mathcal{A}$, aσ – algebra, is the set of all possible coalitions of agents; μ is a probability measure, an indicator of the totality of agents in each coalition of $\mathcal{A}$. If A is finite, then $\mathcal{E}$ is **finite pure exchange economy.**

The variable X denote the triangular fuzzy random variable with parameters mean μ and standard deviation σ respectively. The space $P\{(X_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0 \vee P\{(X_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}$, $(0 \leq \alpha \leq 1)$ is called the **commodity space** so that a point

$$X= P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$$

$\in P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0 \vee (X(a)_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}$ is a **commodity bundle**. Here, $(X, \alpha) = R^2$ is given the task of representing amounts of a specific commodity. In this paper, we confine ourselves to bundles in the non - negative space $P\{(X_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0 \vee (X_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}$.

The measurable fuzzy function from A to $\mathcal{F}(R^2)$ called **consumption fuzzy function**. The fuzzy set $X(a) = P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0 \vee (X(a)_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}$ is called the **consumption fuzzy set** of agent a$\in$ A. It represents all the consumption plans which are a priori possible for a and it may contain positive, zero or negative components. An $(X, \alpha) = R^2$. Here,α - value is always positive $(0 \leq \alpha \leq 1)$ and if the coordinate points of the triangular fuzzy random variables X value is positive. Then$(X, \alpha) = R^2 > 0$is considered to be an input to a. An $(X, \alpha) = R^2$. Here,α - value is always positive $(0 \leq \alpha \leq 1)$, if the coordinate points of the triangular fuzzy random variables X value is negative. Then$(X, \alpha) = R^2 < 0$ is considered to be an output to a. The membership grade $P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0 \vee (X(a)_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}$lies in between [0,1] represents a preference mapping, i.e., a preference relation on commodity space purporting the consumer's tastes.

For two commodity bundles x, y$\in P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0 \vee (X(a)_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}$, we use the following inequality

$$x= P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\} \geq$$

$$y = P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$$

Which means that each components (x, α) of the TRFRV of x $\geq$ each components (y, α) of the TRFRV of y.

The inequality $x = P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\} >$

$$y = P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$$

Which means that

$$P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$$
$$\geq P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\} \text{ and}$$
$$P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$$
$$\neq P\left\{\left(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2\right) \geq 0 \vee \left(Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2\right) \leq 0\right\} \text{and}$$
$$x = P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\} >$$
$$y = P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\}$$

Which means that each components (x, α) of the TRFRV of x > each components (y, α) of the TRFRV of y.

Next, we make reference to the notion of the **price system**. We assume that to every commodity is associated a real number $\pi_i \geq 0$, its price. A vector p = (π_1, π_2) is called a price system. If the price system p prevails, then the real number

$p.P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$ is called the value of the bundle

$P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$.

In this paper we shall always work with non - negatives prices. Every vector of prices p = (π_1, π_2) can be normalized and it is convenient here to let

$P = \{p \in R^2 : 0 < (\pi_1, \pi_2) < 1, (\pi_1 + \pi_2 = 1)\}$.

Let $\bar{P}$ denotes the closure of P and $P_n = \{p \in P: \pi_i \geq \frac{1}{n}, i = 1, 2.\}, n \in \{2, 3\ldots\}$.

To complete the consumption sector of an economy, one introduces the function i: A → $P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee (X(a)_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\}$which assigns to each agent a∈ A the agent's initial endowment vector i(a) = $P\{(i(a)_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\}$. The function "i" is called the **initial allocation**. An agent "a" of the economy $\mathcal{E}$ is fully characterized by the pair

$(X(a), i(a)) = \begin{pmatrix} P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee \{(X(a)_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\}, \\ P\{(i(a)_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\} \end{pmatrix}$. If an agent "a" owns some amount of some commodity, and if the price system p ∈ P prevails, then the function

w(a, p) = $p. P\{(i(a)_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee P\{(i(a)_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\}$, is called **wealth** of the agent a, can be used instead of $P\{(i(a)_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\}$.

In this paper a finite economy (A is finite) is considered. Then, the sum r of all initial allocations $P\{(i(a)_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\}$,

$r = \sum_{a\in A} P\{(i(a)_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\}$, called the **total resources** of $\mathcal{E}$, is an element of$P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee (X(a)_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\}$. The result of exchange activity in a pure exchange economy is a redistribution of the total resources r. Each consumer a∈ A maximizes his satisfaction level by choosing preferable element (an element with higher preference membership grade) from

$P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee (X(a)_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\}$. This leads to exchange of commodities among the members of $\mathcal{E}$ and hence to a redistribution of r – that is, to a new state of economy described by a new allocation f = (f(a): a∈ A). Since only exchange takes place, it is clear that feasibility requires that the condition

$r = \sum_{a\in A} P\{(i(a)^{L}_{\alpha} - (\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)^{U}_{\alpha} + (\alpha-1)\sigma-\mu) \leq 0\}$

$= \sum_{a\in A} f(a)$ must be satisfied.

We call such a f a **feasible allocation**.

For every price system p ∈ $\bar{P}$ and for every a∈ A,

two subsets of $P\{(X(a)^{L}_{\alpha} - (\alpha-1)\sigma-\mu) \geq 0 \vee (X(a)^{U}_{\alpha} + (\alpha-1)\sigma-\mu) \leq 0\}$ are defined: the budget set b(a, p) = $\{P\{(X^{L}_{\alpha} - (\alpha-1)\sigma_1-\mu_1) \geq 0 \vee \{(X^{U}_{\alpha} + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$

$\in P\{(X(a)^{L}_{\alpha} - (\alpha-1)\sigma-\mu) \geq 0 \vee (X(a)^{U}_{\alpha} + (\alpha-1)\sigma-\mu) \leq 0\}$:

$p.P\{(X^{L}_{\alpha} - (\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^{U}_{\alpha} + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \leq w(a, p)\}$ and the demand set

$d(a, p) = \{ P\{(Y^{L}_{\alpha} - (\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y^{U}_{\alpha} + (\alpha-1)\sigma_2-\mu_2) \leq 0\} \in b(p, a)$:

$$P\{(Y(a)^{L}_{\alpha} - (\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y(a)^{U}_{\alpha} + (\alpha-1)\sigma_2-\mu_2) \leq 0\}$$

$\geq P\{(X(a)^{L}_{\alpha} - (\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X(a)^{U}_{\alpha} + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$,

$$P\{(Y^{L}_{\alpha} - (\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y^{U}_{\alpha} + (\alpha-1)\sigma_2-\mu_2) \leq 0\}$$

$\geq P\{(X^{L}_{\alpha} - (\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^{U}_{\alpha} + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\}$, for all $P\{(X^{L}_{\alpha} - (\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^{U}_{\alpha} + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in b(a,p)\}$.

A **competitive equilibrium** for an economy $\mathcal{E}$ is a pair (f, p), where f is a feasible allocation and p is a price system, such that f(a) ∈ d(a, p).

3. Competitive equilibrium for finite pure exchange economy

In this Section, we use the following notions:

$X_{+}(a) = P\{(X(a)^{L}_{\alpha} - (\alpha-1)\sigma-\mu) \geq 0 \vee (X(a)^{U}_{\alpha} + (\alpha-1)\sigma-\mu) \leq 0\}$

$i(a) = P\{(i(a)^{L}_{\alpha} - (\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)^{U}_{\alpha} + (\alpha-1)\sigma-\mu) \leq 0\}$

$x = P\{(X^{L}_{\alpha} - (\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^{U}_{\alpha} + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$ and

$y = P\{(Y^{L}_{\alpha} - (\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y^{U}_{\alpha} + (\alpha-1)\sigma_2-\mu_2) \leq 0\} \in X_{+}(a)$

The following assumptions are useful for this paper.

- $\mathcal{E} = ((A, \mathcal{A}, \mu), X, R^2)$ is a finite pure exchange economy.
- For every $a \in A$,

the set $P\{(X(a)^{L}_{\alpha} - (\alpha-1)\sigma-\mu) \geq 0 \vee (X(a)^{U}_{\alpha} + (\alpha-1)\sigma-\mu) \leq 0\} = R^2_{+}$.

- For every $a \in A$, the set $P\{(X^{L}_{i\alpha} - (\alpha-1)\sigma_i-\mu_i) \geq 0 \vee X^{U}_{i\alpha} + (\alpha-1)\sigma_i-\mu_i) \leq 0\}$

  $\subset P\{(X(a)^{L}_{\alpha} - (\alpha-1)\sigma-\mu) \geq 0 \vee (X(a)^{U}_{\alpha} + (\alpha-1)\sigma-\mu) \leq 0\}$, $0 < \alpha < 1$,

  each $\alpha -$ cuts of the triangular fuzzy random variables are convex.
- For every $a \in A$, for every

$P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$and

$$P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$$

$$\in P\{(X(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (X(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0\}$$

$$P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$$

$< P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$it follows that

$P\{(Y(a)_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y(a)_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$\geq P\{(X(a)_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X(a)_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$.

- If the price system $p \in P$ prevails, then

inf $p.P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} < w(a, p)$.

when $P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$$\in P\{(X(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (X(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0\}$$

- The total resources

  $r = \sum_{a \in A} P\{(i(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0\} > 0$.

Lemma:3.1 Let $a \in A$, $p \in P$ and $c > 0$. Then $d(a, p) = d(a, cp)$.

Proof: If $c > 0$, $x = P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$ and

$y = P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$, from the equivalence

$P \cdot P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$\leq p \cdot P\{(i(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0\} = w(a, p)$

$\Leftrightarrow cp \cdot P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$\leq cp \cdot P\{(i(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0\} = cw(a, p)$

it follows that $b(a, p) = b(a, cp)$, which implies $d(a, p) = d(a, cp)$.

Lemma:3.2 If $p \in P$ and

$P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} \in d(a,p)$.

Then $p \cdot P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} = w(a, p)$, for every $a \in A$.

Proof:

If $P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} \in d(a, p)$, then

$P \cdot P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} \leq w(a, p)$,

$P\{(X(a)_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X(a)_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$\geq P\{(Y(a)_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y(a)_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$and

$P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$$\geq P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$$

forall $P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\} \in b(a, p)$.

Let us assume that

$p\cdot P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\vee(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\neq w(a,p)$,

i.e., $p\cdot P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\vee(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}<w(a,p)$.

Then there exists $P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0\vee(Y_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}\in b(a,p)$,

$$P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\vee(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$$
$$< P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0\vee(Y_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\},$$

Which contradicts the fact that

$$P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\vee(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\in d(a,p).$$

Lemma:3.3 Let $a\in A$ and $p\in P$. Then the budget set b(a, p) is nonempty, compact and convex.

Proof:

Since $P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0\vee(X(a)_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}=R_+^2$ and

Since $\inf p\cdot P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\vee(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}<w(a,p)$,

here, $P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\vee(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$$\in P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0\vee(X(a)_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}$$

the budget set $b(a,p)=\{P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\vee(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}:$

$p\cdot P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\vee(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\leq w(a,p)\}$ is nonempty, bounded, closed and convex.

Lemma:3.4 Let $a\in A$ and $p\in P$. Then the demand set d(a, p) is nonempty, compact and convex.

Proof:

From the lemma 3.3,

the budget set $b(a,p)\in P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0\vee(X(a)_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}$ is a nonempty, compact and convex set and $P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0\vee(X(a)_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}$is upper semi continuous mapping, which means that $P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0\vee(X(a)_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}$attends its maximum on the set b(a, p). Hence $d(a,p)\neq\emptyset$. Further, since d(a, p) is closed subset of the compact set b(a, p), it is compact too.

To prove convexity we shall use the fact that the intersection of convex sets is convex itself. In the lemma3.3, it was proved that b(a, p) is convex. On the other hand, the set d(a,p) is not empty, meaning that there exists$P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\vee(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\in P\{(X(a)_\alpha^L-(\alpha-1)\sigma-\mu)\geq 0\vee(X(a)_\alpha^U+(\alpha-1)\sigma-\mu)\leq 0\}$which belongs to d(a, p).

Each $\alpha-$ cut of the triangular fuzzy random variables are convex.

Then $W=\{P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0\vee(Y_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}\in b(a,p):$

$$P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\vee(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$$
$$\leq P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0\vee(Y_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}\},$$

for all $P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\vee(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\in b(a,p)\}$

$=\{P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0\vee(Y_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}\in b(a,p):$

$P\cdot P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0\vee(Y_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}=w(a,p)\}$ is also convex. Hence, $d(a,p)=P\{(X(a)_{i\alpha}^L-(\alpha-1)\sigma_i-\mu_i)\geq 0\vee(X(a)_{i\alpha}^U+(\alpha-1)\sigma_i-\mu_i)\leq 0\}\cap b(a,p)\cap W$ is a convex set.

Lemma:3.5 If the sequence $\{p_n\}$, n $\in$ N, converges to any point p$\in$ P, then every sequence $\{P(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n) \leq 0\}\}$, n $\in$N , $P(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n) \leq 0\} \in d(a, p_n)$, is bounded.

Proof:

If p = (π_1,π_2), $P_n = (\pi_{n1},\pi_{n2})$, then (since lim n→ ∞$p_n$ = p and $0 < (\pi_1,\pi_2) < 1$), there exists $\in > 0$ and $n_0(\in)$ such that for $n > n_0$,$0 < \min\{\pi_i\} - \in < \pi_{ni} \leq \max\{\pi_i\} + \in <1$,i = 1, 2.Further, if $n > n_0$, for every i = 1, 2. The next implication holds

$P\{(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n) \leq 0\} \in d\ (a, p_n)$

$\Rightarrow P\{(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n) \leq 0\} \in b(a, p_n)$

Since $d\ (a, p_n) \subset b\ (a, p_n)$

$$\Rightarrow p_n. P\{(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n) \leq 0\}$$

$$\leq p_n . P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)^U_\alpha+(\alpha-1)\sigma-\mu) \leq 0\}$$

$$\leq |p_n|. \left|P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)^U_\alpha+(\alpha-1)\sigma-\mu) \leq 0\}\right|$$

$\Rightarrow \rho(X_{ni}, \alpha) < \rho\{(X_{n1},\alpha) + (X_{n2},\alpha)\}$

$$\leq p_n . P\{(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n) \leq 0\}$$

$\Rightarrow (X_{ni},\alpha) < \rho^{-1}|p_n|. \left|P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)^U_\alpha+(\alpha-1)\sigma-\mu) \leq 0\}\right| < M \in R$,

Where $\rho = \min\{\pi_i\} - \in$ and

$P\{(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n) \leq 0\} = (X_n, \alpha)$

$= (X_{n1},\alpha) + (X_{n2},\alpha)$

It means that the sequence $P\{((X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n) \leq 0\}$ is bounded.

Lemma: 3.6

If $\lim_{j\to\infty} p_j$ = p $\in$ P and if $P\{(X^L_{j\alpha}-(\alpha-1)\sigma_j-\mu_j) \geq 0 \vee (X^U_{j\alpha}+(\alpha-1)\sigma_j-\mu_j) \leq 0\} \in d(a, p_j)$, $\lim_{j\to\infty} P\{(X^L_{j\alpha}-(\alpha-1)\sigma_j-\mu_j) \geq 0 \vee (X^U_{j\alpha}+(\alpha-1)\sigma_j-\mu_j) \leq 0\}$

$= P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha+(\alpha-1)\sigma_1-\mu_1) \leq 0\}$,

then $P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha+(\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a, p)$.

Proof:

Since $p_j \cdot P\{(X^L_{j\alpha}-(\alpha-1)\sigma_j-\mu_j) \geq 0 \vee (X^U_{j\alpha}+(\alpha-1)\sigma_j-\mu_j) \leq 0\}$,

$\leq p_j \cdot P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)^U_\alpha+(\alpha-1)\sigma-\mu) \leq 0\}$,

limit $j \to \infty$,we get $p \cdot P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha+(\alpha-1)\sigma_1-\mu_1) \leq 0\}$

$$\leq p \cdot P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)^U_\alpha+(\alpha-1)\sigma-\mu) \leq 0\},$$

which means $P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha+(\alpha-1)\sigma_1-\mu_1) \leq 0\} \in b(a, p)$.

Now to prove that $P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha+(\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a, p)$,

We shall show that

$P\ \{(X(a)^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1) \leq 0\}$

$\geq$ P $\{(Y(a)^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y(a)^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$and

$$P\{(X^L_\alpha - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$$

$$\geq P\{(Y^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$$

For every P $\{(Y^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\} \in b(a, p)$.

We assume that $P\{(i(a)^L_\alpha - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)^U_\alpha + (\alpha-1)\sigma - \mu) \leq 0\} > 0$.

If P $\{(Y^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\} \in b(a, p)$,

Then either p · P $\{(Y^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$$< p \cdot P\{(i(a)^L_\alpha - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)^U_\alpha + (\alpha-1)\sigma - \mu) \leq 0\} \text{ or}$$

p · P $\{(Y^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\} =$

$$p \cdot P\{(i(a)^L_\alpha - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)^U_\alpha + (\alpha-1)\sigma - \mu) \leq 0\}$$

From the inequalityp · P $\{(Y^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$$< p \cdot P\{(i(a)^L_\alpha - (\alpha-1)\sigma - \mu) \geq 0 \vee \{(i(a)^U_\alpha + (\alpha-1)\sigma - \mu) \leq 0\},$$

we get p_n. P $\{(Y^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$< P_n \cdot P\{(i(a)^L_\alpha - (\alpha-1)\sigma - \mu) \geq 0 \vee \{(i(a)^U_\alpha + (\alpha-1)\sigma - \mu) \leq 0\}$,for large $n \in N$,

which means that P $\{(Y^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\} \in b(a, P_n)$

Since $P\{(X^L_{n\alpha} - (\alpha-1)\sigma_n - \mu_n) \geq 0 \vee (X^U_{n\alpha} + (\alpha-1)\sigma_n - \mu_n) \leq 0\} \in d(a, P_n)$,

It is clear that P $\{(X(a)^L_{n\alpha} - (\alpha-1)\sigma_n - \mu_n) \geq 0 \vee (X(a)^U_{n\alpha} + (\alpha-1)\sigma_n - \mu_n) \leq 0\}$

$$\geq P\{(Y(a)^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y(a)^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$$

We know that everyα- cut of the triangular fuzzy numberis closed, bounded and

$$P\{(X^L_{n\alpha} - (\alpha-1)\sigma_n - \mu_n) \geq 0 \vee (X^U_{n\alpha} + (\alpha-1)\sigma_n - \mu_n) \leq 0\}$$

$\in P\{(X(a)^L_{i\alpha} - (\alpha-1)\sigma_i - \mu_i) \geq 0 \vee (X(a)^U_{i\alpha} + (\alpha-1)\sigma_i - \mu_i) \leq 0\}$, for all$i \in N$,

We obtain thatP $\{(X^L_\alpha - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$\in P\{(X(a)^L_{i\alpha} - (\alpha-1)\sigma_i - \mu_i) \geq 0 \vee (X(a)^U_{i\alpha} + (\alpha-1)\sigma_i - \mu_i) \leq 0\}$for all $i \in N$

Therefore, P $\{(X(a)^L_\alpha - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X(a)^U_\alpha + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$\geq$ P $\{(Y(a)^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y(a)^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$.

If p. P $\{(Y^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$$= p. P\{(i(a)^L_\alpha - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)^U_\alpha + (\alpha-1)\sigma - \mu) \leq 0\} \neq 0,$$

then there exists a sequence

$\{P\{(Y^L_{n\alpha} - (\alpha-1)\sigma_n - \mu_n) \geq 0 \vee (Y^U_{n\alpha} + (\alpha-1)\sigma_n - \mu_n) \leq 0\}\}$:

$$P\{(Y^L_{n\alpha} - (\alpha-1)\sigma_n - \mu_n) \geq 0 \vee (Y^U_{n\alpha} + (\alpha-1)\sigma_n - \mu_n) \leq 0\}$$

$$< P\{(Y^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\},$$

$$\lim_{n\to\infty} P\{(Y^L_{n\alpha} - (\alpha-1)\sigma_n - \mu_n) \geq 0 \vee (Y^U_{n\alpha} + (\alpha-1)\sigma_n - \mu_n) \leq 0\}$$

$$= P\{(Y^L_\alpha - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2 - \mu_2) \leq 0\},$$

Then p. $P\{(Y^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$

$<p.\ P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu)\geq 0 \vee (i(a))^U_\alpha+(\alpha-1)\sigma-\mu)\leq 0\}$

But, we take the elements $P\{(Y^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$,

We get $P\{(X(a)^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\geq P\{(Y^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$

If we have$P\{(X(a)^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$< P\{(Y(a)^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y(a)^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$.

We know that each $\alpha-$ cuts are closed in triangular fuzzy random variable,$P\{(Y^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$

$\in P\{(X(a)^L_{i\alpha}-(\alpha-1)\sigma_i-\mu_i)\geq 0 \vee X(a)^U_{i\alpha}+(\alpha-1)\sigma_i-\mu_i)\leq 0\}$ it imply that

$P\{(X(a)^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$< P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$

$\leq P\{(Y(a)^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y(a)^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$

Which is a contradiction for the inequality

$P\{(X(a)^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\geq P\{(Y(a)^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y(a)^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$.

So,$P\{(X(a)^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\geq P\{(Y(a)^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y(a)^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$.

In order to prove that $P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\geq P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$

for all $P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (X^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}\in b(a, p)$,

we suppose opposite (i.e.) that there exists

$P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}\in b(a, p)$,

$P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$< P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$.

Then the next implication holds

$P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$< P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$

$\Rightarrow p\cdot P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$< p\cdot P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$

$\leq p\cdot P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu)\geq 0 \vee (i(a)^U_\alpha+(\alpha-1)\sigma-\mu)\leq 0\} = w\ (a, p)$

$\Rightarrow \exists n\in N: p_n. P\{(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (X^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$

$< p_n$. $P\{(i(a)_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\}$= w (a, p_n)

$\Longrightarrow P\{(X_{n\alpha}^L-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n-\mu_n) \leq 0\} \notin$d(a, p_n),

Which contradicts the fact that

$$P\{(X_{n\alpha}^L-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n-\mu_n) \leq 0\} \in d(a, p_n).$$

Hence, $P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$

$$\geq P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\}$$

for all $P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\} \in$b(a,p).

Hence the proof that$P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in$d(a,p)

Lemma:3.7 The set valued mapping d (a, ·) is upper semi-continuous for every p ∈ P.

Proof:

Mapping d(a, ·): P → $2^{R^2}\setminus\emptyset$ is upper semi-continuous if lim sup d(a, P_n) ⊂ d(a, p) for any p ∈ P and any sequence $\{p_n\}$ converging toward p in P.

Now to prove upper semi-continuity of d(a, ·), we shall show that for every sequence $\{p_n\}$, n ∈ N converging to any point p ∈ P and for every sequence $\{P\{(X_{n\alpha}^L-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n-\mu_n) \leq 0\}\}$, n∈ N,

$$P\{(X_{n\alpha}^L-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n-\mu_n) \leq 0\} \in d(a, p_n),$$

There exists a convergent subsequence

$\{P\{(X_{j\alpha}^L-(\alpha-1)\sigma_j-\mu_j) \geq 0 \vee (X_{j\alpha}^U + (\alpha-1)\sigma_j-\mu_j) \leq 0\}\}$, j ∈ N

$\subset\{P\{(X_{n\alpha}^L-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n-\mu_n) \leq 0\}\}$, n ∈ N such that

$$\lim_{j\to\infty} P\{(X_{j\alpha}^L-(\alpha-1)\sigma_j-\mu_j) \geq 0 \vee (X_{j\alpha}^U + (\alpha-1)\sigma_j-\mu_j) \leq 0\}$$

$=P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in$ d(a, p).

According to lemma 3.5, the sequence

$\{P\{(X_{n\alpha}^L-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n-\mu_n) \leq 0\}\}$ is bounded.

Hence, there exists a convergent subsequence

$\{P\{(X_{j\alpha}^L-(\alpha-1)\sigma_j-\mu_j) \geq 0 \vee (X_{j\alpha}^U + (\alpha-1)\sigma_j-\mu_j) \leq 0\}\}$

$\subset\{P\{(X_{n\alpha}^L-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n-\mu_n) \leq 0\}\}$,

$$\lim_{j\to\infty} P\{(X_{j\alpha}^L-(\alpha-1)\sigma_j-\mu_j) \geq 0 \vee (X_{j\alpha}^U + (\alpha-1)\sigma_j-\mu_j) \leq 0\}$$

$= P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$ and

By lemma 3.6,$P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in$d(a,p).

Hence the proof.

Lemma:3.8 Let P, $\bar{P}$, P_n are sets described in Preliminaries. Then

(i) $P_k \subset P_{k+1} \subset P_{k+2} \ldots$

(ii) $\cup_{n=k}^{\infty} P_n = P \subset \lim_{n\to\infty} p_n = \bar{P}$

(iii) P_n is nonempty, compact, convex set for every n ∈{k, k+1, k+2, ...}.

Proof:

The proof of the lemma 3.1, 3.2 and 3.3 is obvious.

Lemma:3.9

Let P_n is a set described in Preliminaries and we know that each $\alpha -$ cuts in triangular fuzzy random variables are closed and bounded.Then

(i) There exists a bounded set $S_n \subset P\{(X(a)^L_\alpha-(\alpha-1)\sigma-\mu) \geq 0 \vee (X(a)^U_\alpha + (\alpha-1)\sigma-\mu) \leq 0\}$ such that for all p ∈ P_n

$\sum_{a\in A}\{P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\} - P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu)\geq 0 \vee (i(a)^U_\alpha + (\alpha-1)\sigma-\mu) \leq 0\}: P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a,\ p)\} \subset S_n$,

(i.e.)$\sum_{a\in A}\{P\{((X^L_\alpha - i(a)^L_\alpha) -(\alpha-1)(\sigma_1-\sigma) - (\mu_1-\mu)) \geq 0 \vee ((X^U_\alpha - i(a)^U_\alpha) + (\alpha-1)(\sigma_1-\sigma) - (\mu_1-\mu)) \leq 0\}: P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a,p)\} \subset S_n$,

(ii) If p ∈ P_n, then p. $P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2-\mu_2) \leq 0\} = 0$ for every $P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2-\mu_2) \leq 0\} \in$ $\sum_{a\in A}\{P\{((X^L_\alpha - i(a)^L_\alpha) -(\alpha-1)(\sigma_1-\sigma) - (\mu_1-\mu)) \geq 0 \vee ((X^U_\alpha - i(a)^U_\alpha) + (\alpha-1)(\sigma_1-\sigma) - (\mu_1-\mu)) \leq 0\}: P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a,p)\}$.

Proof:

Now to prove the uniform boundedness of the sets of sets $\sum_{a\in A}\{P\{((X^L_\alpha - i(a)^L_\alpha) -(\alpha-1)(\sigma_1-\sigma) - (\mu_1-\mu)) \geq 0 \vee ((X^U_\alpha - i(a)^U_\alpha) + (\alpha-1)(\sigma_1-\sigma) - (\mu_1-\mu)) \leq 0\}: P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a,p)\}$,p∈ p_n

Let us suppose opposite, (i.e.) the bounded region when the $(\alpha = m) -$ cuts (say), m∈(0, 1)There exists $P_m \in P_n$ such that $\sum_{a\in A}\{P\{((X^L_{1\alpha} - i(a)^L_\alpha) -(\alpha-1)(\sigma_1-\sigma) - (\mu_1-\mu)) \geq 0 \vee ((X^U_{1\alpha} - i(a)^U_\alpha) + (\alpha-1)(\sigma_1-\sigma) - (\mu_1-\mu)) \leq 0\}: P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a,\ p_m)\}$does not contain the bounded region when $(\alpha = m) -$ cuts. Since the set A of agents is finite, for some a ∈ A, there exists a sequence $\{P\{(X^L_{i\alpha}-(\alpha-1)\sigma_i-\mu_i) \geq 0 \vee (X^U_{i\alpha} + (\alpha-1)\sigma_i-\mu_i) \leq 0\}\}$, $P\{(X^L_{i\alpha}-(\alpha-1)\sigma_i-\mu_i) \geq 0 \vee (X^U_{i\alpha} + (\alpha-1)\sigma_i-\mu_i) \leq 0\} \in d(a,\ p_i)$, $\{p_i\}\subset\{p_m\}$, such that $\lim_{m\to\infty}\|P\{(X^L_{i\alpha}-(\alpha-1)\sigma_i-\mu_i) \geq 0 \vee (X^U_{i\alpha} + (\alpha-1)\sigma_i-\mu_i) \leq 0\}\| = \infty$.

Since the related sequence $\{P_i\}\subset P_n$ is bounded, there exists a convergent subsequence $\{p_j\}\subset\{p_i\}$, $\lim_{j\to\infty} p_j =$ p∈ $p_n \subset P$. As it was proved in lemma 3.5, for every sequence $\{p_j\}$,j ∈ N, converging to any point p ∈ P and for every sequence $\{P\{(x^L_{j\alpha}-(\alpha-1)\sigma_j-\mu_j)\geq 0 \vee (x^U_{j\alpha} + (\alpha-1)\sigma_j-\mu_j) \leq 0\} \subset\}$, $P\{(x^L_{j\alpha}-(\alpha-1)\sigma_j-\mu_j)\geq 0 \vee (x^U_{j\alpha} + (\alpha-1)\sigma_j-\mu_j) \leq 0\} \in d(a,\ p_j)$, the sequence $\{P\{(x^L_{j\alpha}-(\alpha-1)\sigma_j-\mu_j)\geq 0 \vee (x^U_{j\alpha} + (\alpha-1)\sigma_j-\mu_j) \leq 0\}\}$is bounded.

But $\{P\{(x^L_{j\alpha}-(\alpha-1)\sigma_j-\mu_j) \geq 0 \vee (x^U_{j\alpha} + (\alpha-1)\sigma_j-\mu_j) \leq 0\} \subset$

$\{P\{(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha} + (\alpha-1)\sigma_n-\mu_n) \leq 0\}\}$ and this contradicts the fact that

$\lim_{n\to\infty} \|P\{(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha} + (\alpha-1)\sigma_n-\mu_n) \leq 0\}\| = \infty$, which means that the supposition is not correct.

Now to prove (ii),

Let $P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2-\mu_2) \leq 0\} \in$

$\sum_{a\in A}\{P\{((X^L_\alpha - i(a)^L_\alpha) - (\alpha-1)(\sigma_1 - \sigma) - (\mu_1 - \mu)) \geq 0 \vee ((X^U_\alpha - i(a)^U_\alpha) + (\alpha-1)(\sigma_1 - \sigma) - (\mu_1 - \mu)) \leq$

$0\}: P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a,p)\}$. Then

$$P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2-\mu_2) \leq 0\} = \sum_{a\in A}\{P\{((X(a)^L_\alpha - i(a)^L_\alpha) - (\alpha-1)(\sigma_1 - \sigma) - (\mu_1 - \mu)) \geq 0 \vee ((X(a)^U_\alpha - i(a)^U_\alpha) + (\alpha-1)(\sigma_1 - \sigma) - (\mu_1 - \mu)) \leq 0\}: P\{(X(a)^L_\alpha-(\alpha-1)\sigma-\mu) \geq 0 \vee (X(a)^U_\alpha + (\alpha-1)\sigma-\mu) \leq 0\} \in d(a,p)\}$$

Furtherp$\cdot$ $P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y^U_\alpha + (\alpha-1)\sigma_2-\mu_2) \leq 0\}$

$$=\sum_{a\in A}\left\{\begin{matrix} p\cdot P\{(X(a)^L_\alpha-(\alpha-1)\sigma_a-\mu_a) \geq 0 \vee (X(a)^U_\alpha + (\alpha-1)\sigma_a-\mu_a) \leq 0\} \\ - p\cdot P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)^U_\alpha + (\alpha-1)\sigma-\mu) \leq 0\} \end{matrix}\right\}$$

$= \sum_{a\in A}(w(a,p) - w(a,p)) = 0.$

Lemma: 3.10

Let P, $\bar{P}$ are sets described in Preliminaries. Let $\{p_m\} \subset$ P and $\lim_{m\to\infty} p_m = p \in \frac{\bar{P}}{P}$.

If p$\cdot$ $P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a)^U_\alpha + (\alpha-1)\sigma-\mu) \leq 0\} > 0$, then

$\liminf_{m\to\infty} \{\|P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\|:$

$P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee \{(X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a, p_m)\} = \infty.$

Proof:

First we suppose opposite,

(i.e.) $\liminf_{m\to\infty} \{\|P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\|:$

$P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a, P_m)\}$. Then there exists a bounded set S$\subset$ R such thatd(a, P_m) $\cap$ S $\neq \emptyset$ for infinitely many m's and there exists a bounded sequence

$\{P\{(X^L_{m\alpha}-(\alpha-1)\sigma_m-\mu_m) \geq 0 \vee (X^U_{m\alpha} + (\alpha-1)\sigma_m-\mu_m) \leq 0\}\}, m \in N$

$P\{(X^L_{m\alpha}-(\alpha-1)\sigma_m-\mu_m) \geq 0 \vee (X^U_{m\alpha} + (\alpha-1)\sigma_m-\mu_m) \leq 0\} \in d(a, P_m) \cap S.$

therefore, there exists a convergent subsequence

$$\{P\{(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha} + (\alpha-1)\sigma_n-\mu_n) \leq 0\} \subset \{P\{(X^L_{m\alpha}-(\alpha-1)\sigma_m-\mu_m) \geq 0 \vee (X^U_{m\alpha} + (\alpha-1)\sigma_m-\mu_m) \leq 0\}\}$$

converging to some $P\{(X^L_\alpha-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X^U_\alpha + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$.

Since $P_n P\{(X^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X^U_{n\alpha} + (\alpha-1)\sigma_n-\mu_n) \leq 0\}$

$$\leq P_n \, P\{(i(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0\},$$

let $\lim_{n\to\infty}$, We get p $P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$$\leq p \, P\{(i(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0\},$$

which means $P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} \in b(a, p)$.

In order to prove that

$$P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} \in d(a, p),$$

We shall show that

$$P\{(X(a)_\alpha^L - (\alpha-1)\sigma 1 - \mu 1) \geq 0 \vee (X(a)_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$$

$$\geq \{(Y(a)_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y(a)_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$$

And $P\{(X_\alpha^L - (\alpha-1)\sigma 1 - \mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$$\geq \quad P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$$

for every $P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\} \in b(a, p)$.

Since $P\{(i(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0 > 0$ and

since $P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\} \in b(a, p)$,

then either $p \cdot P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$< p \cdot P\{(i(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0$ or

$p \cdot P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$$= p \cdot P\{(i(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0\},$$

From inequality $p \cdot P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$< p \cdot P\{(i(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0\}$, for large n,

We get $p_n \cdot P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$< p_n \cdot P\{(i(a)_\alpha^L - (\alpha-1)\sigma - \mu) \geq 0 \vee (i(a)_\alpha^U + (\alpha-1)\sigma - \mu) \leq 0\}$, which means that

$P\{(Y_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\} \in b(a, p_n)$.

Since $P\{(X_{n\alpha}^L - (\alpha-1)\sigma_n - \mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n - \mu_n) \leq 0\} \in d(a, p_n)$,

it is clear that

$$P\{(X(a)_{n\alpha}^L - (\alpha-1)\sigma_n - \mu_n) \geq 0 \vee (X(a)_{n\alpha}^U + (\alpha-1)\sigma_n - \mu_n) \leq 0\}$$

$$\geq P\{(Y(a)_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (Y(a)_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$$

here, the $\alpha -$ cuts are closed and bounded in triangular fuzzy random variable.

$$P\{(X_{n\alpha}^L - (\alpha-1)\sigma_n - \mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n - \mu_n) \leq 0\}$$

$$\in P\{(X(a)_{i\alpha}^L - (\alpha-1)\sigma_i - \mu_i) \geq 0 \vee (X(a)_{i\alpha}^U + (\alpha-1)\sigma_i - \mu_i) \leq 0\}$$

We obtain that $P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$$\in P\{(X(a)_{i\alpha}^L - (\alpha-1)\sigma_i - \mu_i) \geq 0 \vee (X(a)_{i\alpha}^U + (\alpha-1)\sigma_i - \mu_i) \leq 0\}$$

Therefore, $P\{(X(a)^L_\alpha-(\alpha-1)\sigma 1-\mu 1)\geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\geq P\{(Y(a)^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y(a)^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$.

If p· $P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$

$=$ p· $P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu)\geq 0 \vee (i(a)^U_\alpha+(\alpha-1)\sigma-\mu)\leq 0\}\neq 0$, then there exists a sequence $\{P\{(Y^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}\}$:

$P\{(Y^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$

$< P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$,

$\lim_{n\to\infty} P\{(Y^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$

$= P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$.

Then p· $P\{(Y^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$

$<$p · $P\{(i(a)^L_\alpha-(\alpha-1)\sigma-\mu)\geq 0 \vee (i(a)^U_\alpha+(\alpha-1)\sigma-\mu)\leq 0\}$.

But for that kind of elements

$P\{(Y^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$,

$P\{(X(a)^L_\alpha-(\alpha-1)\sigma 1-\mu 1)\geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\geq P\{(Y(a)^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y(a)^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$.

If one would have $P\{(X(a)^L_\alpha-(\alpha-1)\sigma 1-\mu 1)\geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$< P\{(Y(a)^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y(a)^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$.

Since$P\{(X(a)^L_{i\alpha}-(\alpha-1)\sigma_i-\mu_i)\geq 0 \vee (X(a)^U_{i\alpha}+(\alpha-1)\sigma_i-\mu_i)\leq 0\}$is closed,

then, for some n ∈ N, $P\{(Y^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$

$\in P\{(X(a)^L_{i\alpha}-(\alpha-1)\sigma_i-\mu_i)\geq 0 \vee (X(a)^U_{i\alpha}+(\alpha-1)\sigma_i-\mu_i)\leq 0\}$

It would imply $P\{(X(a)^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$< P\{(Y(a)^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y(a)^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$

$\leq P\{(Y(a)^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y(a)^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$,

Which would contradict the inequality

$P\{(X(a)^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\geq P\{(Y(a)^L_{n\alpha}-(\alpha-1)\sigma_n-\mu_n)\geq 0 \vee (Y(a)^U_{n\alpha}+(\alpha-1)\sigma_n-\mu_n)\leq 0\}$.

So, $P\{(X(a)^L_\alpha-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (X(a)^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\geq P\{(Y(a)^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y(a)^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$.

In order to prove that

$P\{(X^L_\alpha-(\alpha-1)\sigma 1-\mu 1)\geq 0 \vee (X^U_\alpha+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\geq P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$for all $P\{(Y^L_\alpha-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (Y^U_\alpha+(\alpha-1)\sigma_2-\mu_2)\leq 0\}\in b(a,p)$, we suppose the opposite, (i.e.) that there exists

$P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\} \in b(a, p)$,

$P\{(X_\alpha^L-(\alpha-1)\sigma 1-\mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$

$< P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\}$.

Then the next implication holds

$P\{(X_\alpha^L-(\alpha-1)\sigma 1-\mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$

$< P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\}$.

$\Rightarrow p\cdot P\{(X_\alpha^L-(\alpha-1)\sigma 1-\mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$

$< p\cdot P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\}$

$\leq p\cdot P\{(i(a))_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a))_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\} = w(a, p)$

$\Rightarrow\ \exists\, n \in N: p_n \cdot P\{(X_{n\alpha}^L-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n-\mu_n) \leq 0\}$

$< P_n \cdot P\{(i(a))_\alpha^L-(\alpha-1)\sigma-\mu) \geq 0 \vee (i(a))_\alpha^U + (\alpha-1)\sigma-\mu) \leq 0\} = w(a, p_n)$

$\Rightarrow P\{(X_{n\alpha}^L-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n-\mu_n) \leq 0\} \notin d(a, p_n)$,

Which contradicts the fact that

$P\{(X_{n\alpha}^L-(\alpha-1)\sigma_n-\mu_n) \geq 0 \vee (X_{n\alpha}^U + (\alpha-1)\sigma_n-\mu_n) \leq 0\} \in d(a, p_n)$.

Hence, $P\{(X_\alpha^L-(\alpha-1)\sigma 1-\mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$

$\geq P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\}$

For all $P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\} \in b(a, p)$.

It completes the proof that

$P\{(X_\alpha^L-(\alpha-1)\sigma 1-\mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a, p)$.

On the other hand, Since $p \in \frac{\bar{P}}{P}$, it flows that b(a, p) is an unbounded set.

But if$P\{(X_\alpha^L-(\alpha-1)\sigma 1-\mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a, p)$ then

$$P\{(X_\alpha^L-(\alpha-1)\sigma 1-\mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$$

$$\geq P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\}$$

for all $P\{(Y_\alpha^L-(\alpha-1)\sigma_2-\mu_2) \geq 0 \vee (Y_\alpha^U + (\alpha-1)\sigma_2-\mu_2) \leq 0\} \in b(a, p)$, which means that the set d(a, p) is empty.

This contradicts the existence of the limit $P\{(X_\alpha^L-(\alpha-1)\sigma 1-\mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a, p)$,

It means that the supposition that

$\liminf_{m\to\infty} \{\|P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\|:$

$P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\} \in d(a, P_m)\}$ is not correct.

For the set S_n, n $\in$\{k, k+1, k+2, ...\}, from the Lemma 3.9, part (i), we can choose compact set S_n. Then the convex hull of S (denoted by conS_n), is convex, compact set with the same properties. The set valued mapping F_n: con$S_n \to 2^{P_n}$ is defined by$F_n(P\{(X_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$

$= \{p \in P_n: p\cdot P\{(X_\alpha^L-(\alpha-1)\sigma 1-\mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}$

$=\max_{q \in P_n} q \cdot P\{(X_\alpha^L - (\alpha-1)\sigma 1 - \mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}\}$

Lemma: 3.11

The mapping $F_n(P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$= \{p \in P_n : p \cdot P\{(X_\alpha^L - (\alpha-1)\sigma 1 - \mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$= \max_{q \in P_n} q \cdot P\{(X_\alpha^L - (\alpha-1)\sigma 1 - \mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}\}.$

has the following properties:

(i) For every $P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} \in \mathrm{con} S_n$, $F_n(P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}) \subset P_n$ is nonempty, compact,convex set.

(ii) $F_n$has a closed.

Proof:

SinceP_n is a compact set and since scalar product is a continuous operation, it attends maximum on P_n, which means that $F_n(P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} \neq 0$ and it is closed subset of the compact set P_n, thus it is compact. To prove convexity, consider $(\mu_1 - \sigma_1)$, $(\mu_1 + \sigma_1)$

$\in F_n(P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\})$.

From the definition of F_n,

we get $(\mu_1 - \sigma_1)P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$= \max_{q \in P_n} q \cdot P\{(X_\alpha^L - (\alpha-1)\sigma 1 - \mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$= m$

And$(\mu_1 + \sigma_1)P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$= \max_{q \in P_n} q \cdot P\{(X_\alpha^L - (\alpha-1)\sigma 1 - \mu 1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} = m$

Then

$(b(\mu_1 - \sigma_1) + (1-b)(\mu_1 + \sigma_1))\ P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$= P\{(bR_\alpha^L - (\alpha-1)b\sigma_1 - b\mu_1) \geq 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\} = m,$

(i.e.)$(b(\mu_1 - \sigma_1) + (1-b)(\mu_1 + \sigma_1))$

$\in F_n(P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\})$.

Now to prove 2, we have to show that,

if $(P\{(X_{i\alpha}^L - (\alpha-1)\sigma_i - \mu_i) \geq 0 \vee (X_{i\alpha}^U + (\alpha-1)\sigma_i - \mu_i) \leq 0\}, p_i) \in \mathrm{gr}\ F$,

$\lim_{i \to \infty} P\{(X_{i\alpha}^L - (\alpha-1)\sigma_i - \mu_i) \geq 0 \vee (X_{i\alpha}^U + (\alpha-1)\sigma_i - \mu_i) \leq 0\}$

$= P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\},$

$\lim_{i \to \infty} p_i = p$, then $p \in F_n(P\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\})$.

If $(P\{(X_{i\alpha}^L - (\alpha-1)\sigma_i - \mu_i) \geq 0 \vee (X_{i\alpha}^U + (\alpha-1)\sigma_i - \mu_i) \leq 0\}, p_i) \in \mathrm{gr}\ F$, then

$P\{(X_{i\alpha}^L - (\alpha-1)\sigma_i - \mu_i) \geq 0 \vee (X_{i\alpha}^U + (\alpha-1)\sigma_i - \mu_i) \leq 0\} \cdot q$

$\leq P\{(X_{i\alpha}^{L} - (\alpha-1)\sigma_i - \mu_i) \geq 0 \vee (X_{i\alpha}^{U} + (\alpha-1)\sigma_i - \mu_i) \leq 0\} \cdot P_i$ for every $q \in p_n$.

From continuity of scalar product, letting $i \to \infty$, for every $q \in p_n$

we have $\lim_{i\to\infty} (P\{(X_{i\alpha}^{L} - (\alpha-1)\sigma_i - \mu_i) \geq 0 \vee (X_{i\alpha}^{U} + (\alpha-1)\sigma_i - \mu_i) \leq 0\} \cdot q$

$$\leq P\{(X_{i\alpha}^{L} - (\alpha-1)\sigma_i - \mu_i) \geq 0 \vee (X_{i\alpha}^{U} + (\alpha-1)\sigma_i - \mu_i) \leq 0\}, p_i)$$

$\Longrightarrow P\{(X_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_{\alpha}^{U} + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} \cdot q$

$$\leq P\{(X_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee \left(X_{\alpha}^{U} + (\alpha-1)\sigma_1 - \mu_1\right) \leq 0\} \cdot p.$$

So, we have proved that

$p \in F_n(P\{(X_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (X_{\alpha}^{U} + (\alpha-1)\sigma_1 - \mu_1) \leq 0\})$.

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