On Hessenberg of Triangular Fuzzy Number Matrices

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ABSTRACT

The fuzzy set theory has been applied in many fields such as management, engineering, theory of matrices and so on. In this paper, some elementary operations on proposed triangular fuzzy numbers (TFNs) are defined. We also have been defined some operations on triangular fuzzy matrices(TFMs). The notion of Hessenberg fuzzy matrices are introduced and discussed. Some of their relevant properties have also been verified

Keywords - *Fuzzy Arithmetic, Fuzzy number, Triangular fuzzy number (TFN), Triangular fuzzy matrix(TFM), Hessenberg fuzzy matrix(HFM).*

I. INTRODUCTION

Fuzzy sets have been introduced by Lofti.A.Zadeh[12] Fuzzy set theory permits the gradual assessments of the membership of elements in a set which is described in the interval [0,1]. It can be used in a wide range of domains where information is incomplete and imprecise. Interval arithmetic was first suggested by Dwyer [2] in 1951, by means of Zadeh's extension principle [13,14], the usual Arithmetic operations on real numbers can be extended to the ones defined on Fuzzy numbers. Dubosis and Prade [1] has defined any of the fuzzy numbers as a fuzzy subset of the real line [4]. A fuzzy number is a quantity whose values are imprecise, rather than exact as is the case with single – valued numbers.

Triangular fuzzy numbers (TFNs) are frequently used in application. It is well known that the matrix formulation of a mathematical formula gives extra facility to study the problem. Due to the presence of uncertainty in many mathematical formulations in different branches of science and technology. We introduce triangular fuzzy matrices (TFMs). To the best of our knowledge, no work is available on TFMs, through a lot of work on fuzzy matrices is available in literature. A brief review on fuzzy matrices is given below.

Fuzzy matrices were introduced for the first time by Thomason [11] who discussed the convergence of power of fuzzy matrix. Fuzzy matrices play an important role in scientific development. Two new operations and some applications of fuzzy matrices are given in [7,8,9,10]. Hessenberg matrices play an important role in many application and have been the object of several studies [3,5,6].

The paper organized as follows, Firstly in section 2, we recall the definition of Triangular fuzzy number and some operations on triangular fuzzy numbers (TFNs). In section 3, we have reviewed the definition of triangular fuzzy matrix (TFM) and some operations on Triangular fuzzy matrices (TFMs). In section 4, we defined the notion of Hessenberg triangular fuzzy matrices. In section 5, we have presented some properties of Hessenberg of Triangular fuzzy matrices. Finally in section 6, conclusion is included.

II. PRELIMINARIES

In this section, We recapitulate some underlying definitions and basic results of fuzzy numbers.

Definition 2.1 Fuzzy set

A fuzzy set is characterized by a membership function mapping the element of a domain, space or universe of discourse X to the unit interval [0,1]. A fuzzy set A in a universe of discourse X is defined as the following set of pairs

$$A=\{(x,\mu_A(x))\ ;\ x\in X\}$$

Here $\mu_A : X \to [0,1]$ is a mapping called the degree of membership function of the fuzzy set A and $\mu_A(x)$ is called the membership value of $x\in X$ in the fuzzy set A.

These membership grades are often represented by real numbers ranging from [0,1].

Definition 2.2 Normal fuzzy set

A fuzzy set A of the universe of discourse X is called a normal fuzzy set implying that there exists at least one x∈X such that μ_A(x) =1.

Definition 2.3 Convex fuzzy set

A fuzzy set A={(x,μ_A(x))}⊆X is called Convex fuzzy set if all $A_\propto$ are Convex set (i.e.,) for every element $x_1 \in A_\propto$ and $x_2 \in A_\propto$ for every α∈[0,1], λx_1+(1-λ) $x_2 \in A_\alpha$ for all λ∈[0,1] otherwise the fuzzy set is called non-convex fuzzy set.

Definition 2.4 Fuzzy number

A fuzzy set $\tilde{A}$ defined on the set of real number R is said to be fuzzy number if its membership function has the following characteristics

i. $\tilde{A}$ is normal
ii. $\tilde{A}$ is convex
iii. The support of $\tilde{A}$ is closed and bounded then $\tilde{A}$ is called fuzzy number.

Definition 2.5 Triangular fuzzy number

A fuzzy number $\tilde{A}$ = (a_1, a_2, a_3) is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & ; x \leq a_1 \\ \dfrac{x-a_1}{a_2-a_1} & ; a_1 \leq x \leq a_2 \\ 1 & ; x = a_2 \\ \dfrac{a_3-x}{a_3-a_2} & ; a_2 \leq x \leq a_3 \\ 0 & ; x > a_3 \end{cases}$$

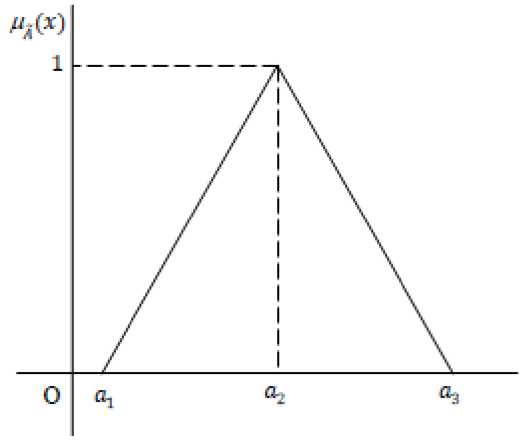

Fig:1 Triangular Fuzzy Number

Definition 2.6 Ranking function

We defined a ranking function ℜ: F(R)→R which maps each fuzzy numbers to real line F(R) represent the set of all triangular fuzzy number. If R be any linear ranking function

$$\Re(\tilde{A}) = \left(\frac{a_1+a_2+a_3}{3}\right)$$

Also we defined orders on F(R) by

(i). ℜ ($\tilde{A}$) ≥ ℜ ($\tilde{B}$) if and only if $\tilde{A} \geq_R \tilde{B}$

(ii). ℜ ($\tilde{A}$) ≤ ℜ ($\tilde{B}$) if and only if $\tilde{A} \leq_R \tilde{B}$

(iii). ℜ ($\tilde{A}$) = ℜ ($\tilde{B}$) if and only if $\tilde{A} =_R \tilde{B}$

Definition 2.7 Arithmetic operations on triangular fuzzy numbers (TFNs)

Let $\tilde{A}$ = (a_1, a_2, a_3) and $\tilde{B}$=(b_1, b_2, b_3) be triangular fuzzy numbers (TFNs) then we defined,

Addition

$\tilde{A}+\tilde{B} = (a_1+b_1, a_2+b_2, a_3+b_3)$

Subtraction

$\tilde{A}-\tilde{B} = (a_1-b_3, a_2-b_2, a_3-b_1)$

Multiplication

$\tilde{A}\times\tilde{B}$ =(a_1ℜ(B), a_2ℜ(B), a_3ℜ(B))

where $\Re(\tilde{B})=\left(\frac{b_1+b_2+b_3}{3}\right)$ or $\Re(\tilde{b})=\left(\frac{b_1+b_2+b_3}{3}\right)$

Division

$\tilde{A} \div \tilde{B} = \left(\frac{a_1}{\Re(\tilde{B})}, \frac{c}{\Re(\tilde{B})}, \frac{a_3}{\Re(\tilde{B})}\right)$

where $\Re(\tilde{B}) = \left(\frac{b_1+b_2+b_3}{3}\right)$ or $\Re(\tilde{b}) = \left(\frac{b_1+b_2+b_3}{3}\right)$

Scalar multiplication

$$k\tilde{A} = \begin{cases} (ka_1, ka_2, ka_3) & if\ k \geq 0 \\ (ka_3, ka_2, ka_1) & if\ k < 0 \end{cases}$$

Definition 2.8 Zero triangular fuzzy number

If $\tilde{A}$ = (0,0,0) then $\tilde{A}$ is said to be zero triangular fuzzy number. It is defined by 0.

Definition 2.9 Zero equivalent triangular fuzzy number

A triangular fuzzy number $\tilde{A}$ is said to be a zero equivalent triangular fuzzy number if $\Re(\tilde{A})$=0. It is defined by $\tilde{0}$.

Definition 2.10 Unit triangular fuzzy number

If $\tilde{A}$ = (1,1,1) then $\tilde{A}$ is said to be a unit triangular fuzzy number. It is denoted by 1.

Definition 2.11 Unit equivalent triangular fuzzy number

A triangular fuzzy number $\tilde{A}$ is said to be unit equivalent triangular fuzzy number. If $\Re(\tilde{A}) = 1$. It is denoted by $\tilde{1}$.

Definition 2.12 Inverse of triangular fuzzy number

If $\tilde{a}$ is triangular fuzzy number and $\tilde{a} \neq \tilde{0}$ then we define. $\tilde{a}^{-1} = \frac{\tilde{1}}{\tilde{a}}$.

III. TRIANGULAR FUZZY MATRICES (TFMs)

In this section, we introduced the triangular fuzzy matrix and the operations of the matrices some examples provided using the operations.

Definition 3.1 Triangular fuzzy matrix (TFM)

A triangular fuzzy matrix of order m×n is defined as A= $(\tilde{a}_{ij})_{m\times n}$, where $a_{ij} = (a_{ij1}, a_{ij2}, a_{ij3})$ is the ij^{th} element of A.

Definition 3.2 Operations on Triangular Fuzzy Matrices (TFMs)

As for classical matrices. We define the following operations on triangular fuzzy matrices. Let A = $(\tilde{a}_{ij})$ and B = $(\tilde{b}_{ij})$ be two triangular fuzzy matrices (TFMs) of same order. Then, we have the following

i A+B = $(\tilde{a}_{ij} + \tilde{b}_{ij})$

ii A-B = $(\tilde{a}_{ij} - \tilde{b}_{ij})$

iii For A = $(\tilde{a}_{ij})_{m\times n}$ and B = $(\tilde{b}_{ij})_{n\times k}$ then AB = $(C_{ij})_{m\times k}$ where $\tilde{C}_{ij} = \sum_{p=1}^{n} \tilde{a}_{ip} \cdot \tilde{b}_{pj}$, i=1,2,…m and j=1,2,…k

iv A^T or $A^1 = (\tilde{a}_{ji})$

v $KA = (K\tilde{a}_{ij})$ where K is scalar.

Examples

(1).If$A = \begin{bmatrix} (1,2,3) & (2,4,6) \\ (-1,1,3) & (2,3,4) \end{bmatrix}$ and B= $\begin{bmatrix} (2,3,4) & (3,5,7) \\ (4,5,6) & (4,5,9) \end{bmatrix}$

Then, A+B=$(\tilde{a}_{ij} + \tilde{b}_{ij})$

$$A + B = \begin{bmatrix} (1,2,3) & (2,4,6) \\ (-1,1,3) & (2,3,4) \end{bmatrix} + \begin{bmatrix} (2,3,4) & (3,5,7) \\ (4,5,6) & (4,5,9) \end{bmatrix}$$

$$A + B = \begin{bmatrix} (3,5,7) & (5,9,13) \\ (3,6,9) & (6,8,13) \end{bmatrix}.$$

(2).If$A = \begin{bmatrix} (1,2,3) & (2,4,6) \\ (-1,1,3) & (2,3,4) \end{bmatrix}$ and B= $\begin{bmatrix} (2,3,4) & (3,5,7) \\ (4,5,6) & (4,5,9) \end{bmatrix}$

Then, A-B=$(\tilde{a}_{ij} - \tilde{b}_{ij})$

$$A - B = \begin{bmatrix} (1,2,3) & (2,4,6) \\ (-1,1,3) & (2,3,4) \end{bmatrix} - \begin{bmatrix} (2,3,4) & (3,5,7) \\ (4,5,6) & (4,5,9) \end{bmatrix}$$

$$A - B = \begin{bmatrix} (-3,-1,1) & (-5,-1,3) \\ (-7,-4,-1) & (-7,-2,0) \end{bmatrix}.$$

(3). If $A = \begin{bmatrix} (1,2,3) & (2,4,6) \\ (-1,1,3) & (2,3,4) \end{bmatrix}$and B= $\begin{bmatrix} (2,3,4) & (3,5,7) \\ (4,5,6) & (4,5,9) \end{bmatrix}$

Then, $A.B = (\tilde{a}_{ij}\tilde{b}_{ij})$

$$A.B = \begin{bmatrix} (1,2,3) & (2,4,6) \\ (-1,1,3) & (2,3,4) \end{bmatrix} . \begin{bmatrix} (2,3,4) & (3,5,7) \\ (4,5,6) & (4,5,9) \end{bmatrix}$$

$A.B$ =
$$\begin{bmatrix} (1,2,3)(3)+(2,4,6)(5) & (1,2,3)(5)+(2,4,6)(6) \\ (-1,1,3)(3)+(2,3,4)(5) & (-1,1,3)(5)+(2,3,4)(6) \end{bmatrix}$$

$$A.B=\begin{bmatrix} (13,26,39) & (17,34,51) \\ (7,18,29) & (7,23,39) \end{bmatrix}.$$

IV. HESSENBERG TRIANGUAR FUZZY MATRIX (HTFM)

In this section, we introduce the new matrix namely Hessenberg matrix in the fuzzy nature.

Definition 4.1 Lower Hessenberg Fuzzy Matrix

A Square triangular fuzzy matrix $A = (\tilde{a}_{ij})$ is called Lower Hessenberg triangular fuzzy matrix if all the entries above the first super diagonal are zero.

i.e. $\tilde{a}_{ij} = 0; i+1 < j \ \forall \ i,j = 1,2,\dots,n$

Definition 4.2 Upper Hessenberg Fuzzy Matrix

A Square triangular fuzzy matrix $A = (\tilde{a}_{ij})$ is called an upper Hessenberg triangular fuzzy matrix if all the entries below the first sub diagonal are zero.

i.e. $\tilde{a}_{ij} = 0; i > j+1 \ \forall \ i,j = 1,2,\dots,n$

Definition 4.3 Hessenberg Triangular Fuzzy Matrix

A Square triangular fuzzy matrix $A = (\tilde{a}_{ij})$ is called Hessenberg triangular fuzzy matrix (HTFM). If it is either upper Hessenberg triangular fuzzy matrix and lower Hessenberg triangular fuzzy matrix.

Definition 4.4 Lower Hessenberg Equivalent Triangular Fuzzy Matrix

A Square triangular fuzzy matrix $A = (\tilde{a}_{ij})$ is called lower Hessenberg equivalent triangular fuzzy matrix if all the entries above the first super diagonal are $\tilde{0}$

Definition 4.5 Upper Hessenberg Equivalent Triangular Fuzzy Matrix

A Square triangular fuzzy matrix $A = (\tilde{a}_{ij})$ is called Upper Hessenberg equivalent triangular fuzzy matrix. If all the entries below the first super diagonal are $\tilde{0}$

Definition 4.6 Hessenberg Equivalent Triangular Fuzzy Matrix

A Square triangular fuzzy matrix $A = (\tilde{a}_{ij})$ is called Hessenberg- equivalent triangular fuzzy matrix. If it is either upper Hessenberg equivalent triangular fuzzy matrix or lower Hessenberg- equivalent triangular fuzzy matrix.

V. SOME PROPERTIES OF HESSENBERG TRIANGULAR FUZZY MATRICES

In this section, we introduced the properties of Hessenberg Triangular fuzzy Matrices(HTFMs).

5.1. Properties of HTFM (Hessenberg Triangular Fuzzy Matrix)

Property 5.1.1:

The sum of two lower HTFM's of order n is also a lower HTFM of order n.

Proof:

Let $A = (\tilde{a}_{ij})$ and $B = (\tilde{b}_{ij})$ be two lower HTFM's

Since A and B are lower HTFM's then,

$\tilde{a}_{ij} = 0$ and $\tilde{b}_{ij} = 0$, if $i+1 < j$, for all $i,j = 1,\dots.n$.

Let A+B=C then, $(\tilde{a}_{ij} + \tilde{b}_{ij}) = (\tilde{c}_{ij})$.

Since $i+1 < j; i,j = 1,\dots.n$ then,

$$\tilde{c}_{ij} = \tilde{a}_{ij} + \tilde{b}_{ij}$$
$$= 0$$

Hence C is also a lower HTFM of order n.

Property 5.1.2:

The sum of two Upper HTFM's of order n is also an upper HTFM of order n.

Proof:

Let $A = (\tilde{a}_{ij})$ and $B = (\tilde{b}_{ij})$ be two upper HTFM's

Since A and B are upper HTFM's.

Then, $\tilde{a}_{ij} = 0$ and $\tilde{b}_{ij} = 0 \ \forall \ i > j + 1; \quad i, j = 1,2, \dots n$

Let $A + B = C$ then $\left(\tilde{a}_{ij} + \tilde{b}_{ij}\right) = (\tilde{c}_{ij})$

Since $i > j + 1; \quad i, j = 1,2, \dots n$ then,

$$\tilde{c}_{ij} = \tilde{a}_{ij} + \tilde{b}_{ij}$$

$$= 0$$

Hence C is also an upper HTFM of order n.

Property 5.1.3:

The product of lower HTFM by Constant is also a lower HTFM.

Proof:

Let $A = (\tilde{a}_{ij})$ be a lower HTFM.

Since A is lower HTFM, $\tilde{a}_{ij} = 0$, $i + 1 < j$; for all $i, j = 1,2, \dots n$

Let k be a scalar and $kA = B$, then $\left(k\tilde{a}_{ij}\right) = (\tilde{b}_{ij})$.

Since $i + 1 < j; i, j = 1,2..n$ then,

$$\tilde{b}_{ij} = k(\tilde{a}_{ij})$$

$$= k(0)$$

$$= 0$$

Hence B is also a lower HTFM.

Property 5.1.4:

The Product of an upper HTFM by a constant is also an upper HTFM.

Proof:

Let $A = (\tilde{a}_{ij})$ be an upper HTFM.

Since A is an upper HTFM $\tilde{a}_{ij} = 0$,for $i > j + 1$;

$i, j = 1,2 \dots \dots . . n$.

Let k be a scalar and $kA = B$, then $k\tilde{a}_{ij} = \tilde{b}_{ij}$.

Since $i > j + 1; i, j = 1,2 \dots \dots . n$ then,

$$\tilde{b}_{ij} = k(\tilde{a}_{ij}),$$

$$= k(0),$$

$$= 0.$$

Hence B is also an upper HTFM.

Property 5.1.5:

The Product of two lower HTFM of order n is also a lower HTFM of order n.

Proof:

Let $A = (\tilde{a}_{ij})$ and $B = (\tilde{b}_{ij})$ be two lower HTFM's.

Since A and B are lower HTFM then, $\tilde{a}_{ij} = 0$ *and*

$\tilde{b}_{ij} = 0$,$i + 1 < j; for\ all\ i, j = 1,2 \dots n$

Let $AB = C = \tilde{c}_{ij}$

Where $\tilde{c}_{ij} = \sum_{k=1}^{n} \tilde{a}_{ik} . \tilde{b}_{kj}$

We will show that $\tilde{c}_{ij} = 0$,for $i + 1 < j; i, j = 1,2 \dots n$

For $i + 1 < j$

We have $\tilde{a}_{ik} = 0$ for $k = i + 2, i + 3 \dots \dots \dots n$ and $\tilde{b}_{kj} = 0$ for $k = 1,2 \dots \dots . i + 1$

Therefore $\tilde{c}_{ij} = \sum_{k=1}^{n} \tilde{a}_{ik} . \tilde{b}_{kj}$

$$=\left[\sum_{k=1}^{i+1} \tilde{a}_{ik} . \tilde{b}_{kj}\right] + \left[\sum_{k=i+2}^{n} \tilde{a}_{ik} . \tilde{b}_{kj}\right] = 0.$$

Hence C is also lower Hessenberg TFM.

Property 5.1.6:

The Product of two upper HTFMs of order n is also an upper HTFM of order n .

Proof:

Let A=$\tilde{a}_{ij}$ and $B = \tilde{b}_{ij}$ be two upper HTFM.

Since A and B are upper HTFMs.

$\tilde{a}_{ij} = 0$ and $\tilde{b}_{ij} = 0$ for all $i > j + 1; i, j = 1,2 \dots n$.

For $i > j + 1$ we have $\tilde{a}_{ik} = 0$ for $k = 1,2, \dots i$ and Similarly $\tilde{b}_{kj} = 0$ for $k = i + 1 \dots n$.

Therefore $\tilde{c}_{ij} = \sum_{k=1}^{n} \tilde{a}_{ik} . \tilde{b}_{kj}$

$=\sum_{k=1}^{i} \tilde{a}_{ik} . \tilde{b}_{kj} + \sum_{k=i+1}^{n} \tilde{a}_{ik} . \tilde{b}_{kj}$

$= 0$

Hence C is also upper hessenberg TFM.

Property 5.1.7:

The transpose of an upper HTFM is a lower HTFM and vice versa.

Proof:

Let $A = (\tilde{a}_{ij})$ be an upper HTFM.

Since A is an upper HTFM, $\tilde{a}_{ij} = 0$ for all $,i > j + 1; i, j = 1,2, \dots, n$

Let B be the transpose of A then $A' = B$ i.e. $(\tilde{a}_{ji}) = (\tilde{b}_{ij})$

For all $i > j + 1; i, j = 1,2, \dots, n, \tilde{a}_{ij} = 0 = \tilde{b}_{ji}$.

That is for all $i + 1 < j; i, j = 1,2, \dots, n$

$\tilde{b}_{ij} = 0$

Hence B is a lower HTFM.

VI. CONCLUSION

In this article, Hessenberg of triangular fuzzy matrices are defined and some relevant properties of their matrices have also been proved. Few illustrations based on operations of triangular fuzzy matrices have also been justified. In future, these matrices will be apply in the polynomials, generalized fibonacci numbers, and special kind of composition of natural numbers in the domain of fuzzy environment

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